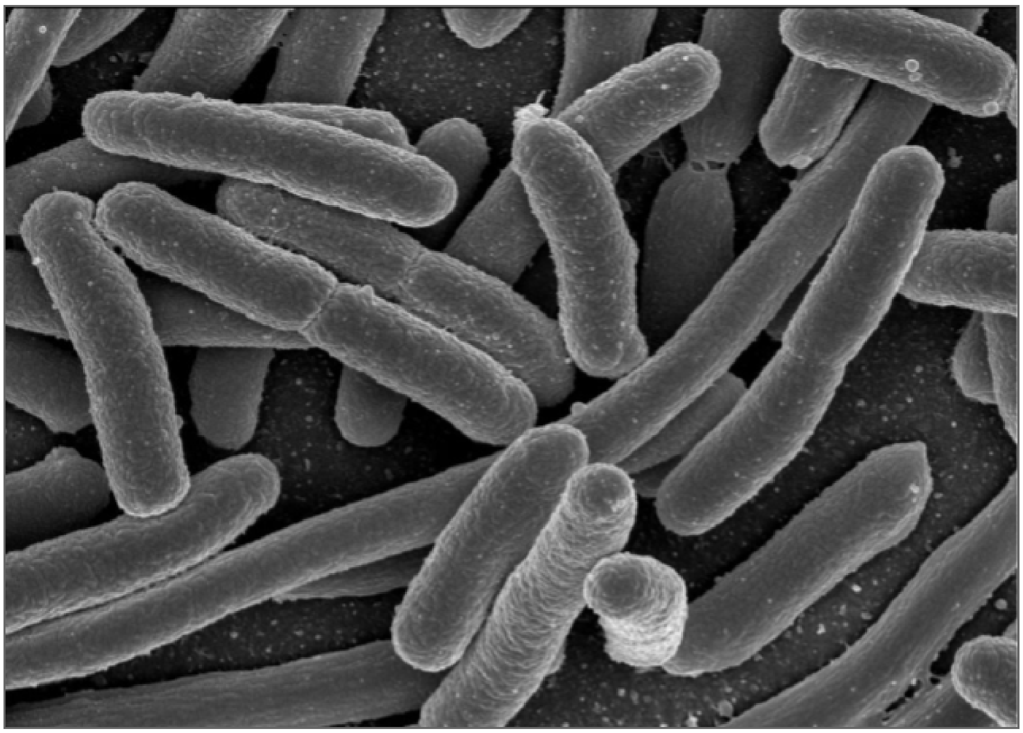
Tools of Algebra: Nonlinear Functions

Fast Track GRASP Math Packet

Part 2



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THE CITY UNIVERSITY OF NEW YORK

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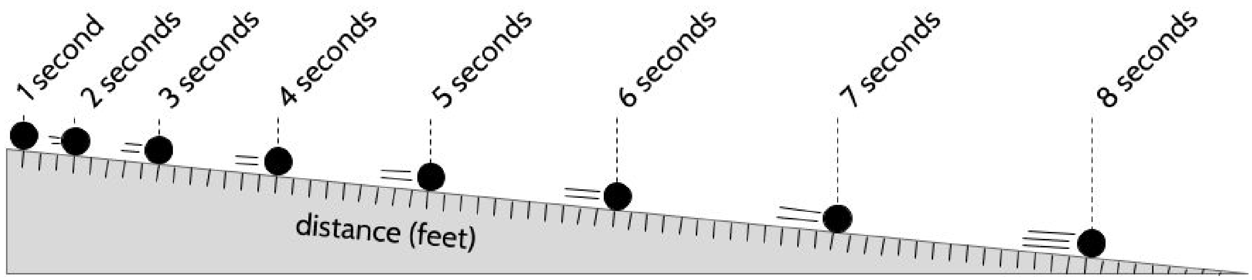
Exploring Quadratic Growth II

Investigating Gravity II

In one of his experiments, Galileo Galilei rolled balls down ramps and measured the distance traveled. He was trying to understand how gravity affects falling objects.



Galileo noticed a pattern in the change in distance. He saw that the change in distance each second was related to the odd numbers (1, 3, 5, 7, 9, 11, 13, 15, etc.). He called this pattern the Law of Odd Numbers.

Galileo saw more patterns by analyzing the change in speed, which is also called *acceleration*. The change from 1 feet per second to 3 feet per second is an acceleration of 2 feet per second squared.

1) Find the change in speed (acceleration) for the rest of the table on the right. For example, what is the change from 3 to 5?

2) What do you notice?

time (seconds)	distance (feet)
0	0
1	1
2	4
3	9
4	16
5	25
6	36
7	49
8	64

speed (change in distance): + 1, + 3, + 5, + 7, + 9, + 11, + 13, + 15

acceleration (change in speed): + 2

The acceleration of the ball rolling down this ramp is 2 feet per second per second. This means that every second, the speed of the ball increases by 2 feet per second. The speed is not constant since it changes continually, but the acceleration of the rolling ball is constant. The change in speed stays the same as the ball rolls down hit. It is always accelerating at a rate of 2 ft/s^2.

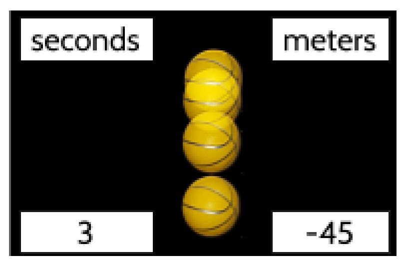
With this experiment, Galileo discovered that acceleration due to gravity is a constant force. A rolling ball on an inclined plane has less acceleration than a ball falling in the air, but both the acceleration of the ball and the acceleration of falling objects are constant. Galileo used this experiment to understand how objects fall freely in the air.

Gravity and Acceleration

The longest pedestrian suspension bridge in the United States opened in Gatlinburg, Tennessee in 2019. It is 680 feet (207 meters) long and about 140 feet (43 meters) high.

Let's imagine that we decide to drop a basketball off the bridge and time how long it takes to hit the ground below.

Warning! Please don't try this at home. This is just a word problem. 🙂



A friend stands on the ground below the bridge with a stopwatch and starts timing when you drop the ball off the top of the bridge. The ball hits the ground after 3 seconds. We'll call the distance -45 meters, since the ball is falling from the bridge that is approximately 45 meters high. If the ball were rising, we would use a positive number for distance.

3) What questions could you ask about the situation at this point?

Some questions occurred to us:

- What is the speed of the ball as it falls toward the ground?
- How far did the ball fall in 1 second?
- How far would the ball fall in 20 seconds?
- What does the graph of this data look like?
- How long would the ball take to hit the ground if we dropped it out of an airplane?

By the way, we are ignoring other factors that affect the speed of the basketball as it falls. For example, we are pretending that there is no air resistance or wind.

In **Tools of Algebra: Expressions, Equations, & Inequalities**, we practiced using a formula for speed, $r = \frac{d}{t}$, where r is speed (rate), d is distance, and t is time.

It seems that we should be able to find the speed of the falling basketball because we know d is -45 meters and t is 3 seconds.

$$r = \frac{-45\ meters}{3\ seconds} = \ -15\ meters\ per\ second\ (mps)$$

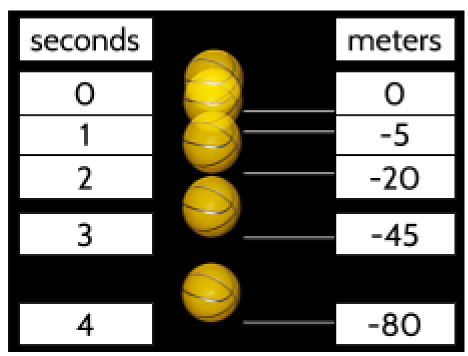
So, the basketball traveled at a speed of -15 mps? The negative sign means the ball fell from a starting point of 0 meters on the bridge.

Could we predict how far the ball would fall after 4 seconds? If the ball is moving at a speed of 15 mps, it seems that it would travel 60 meters in 4 seconds.

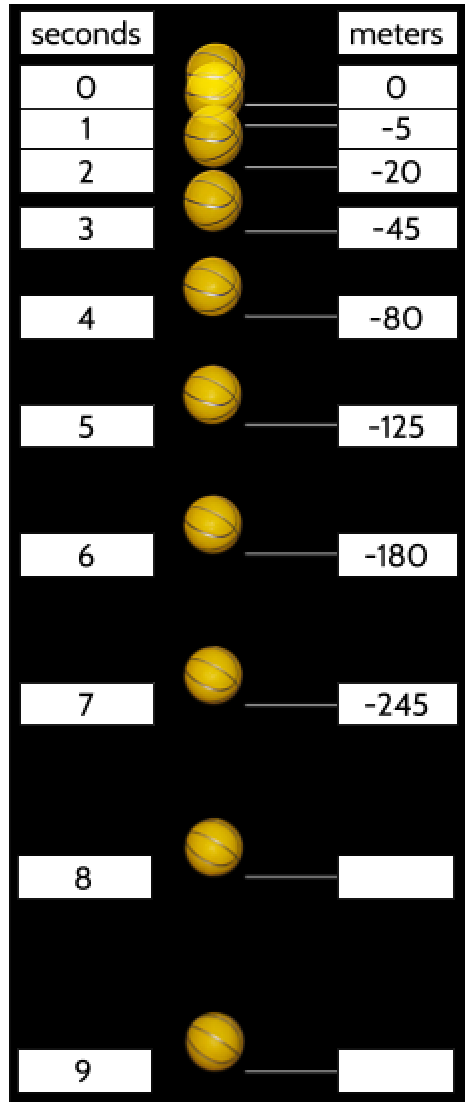
$$-15\ meters\ per\ second\ (mps) \cdot 4\ seconds = \ -60\ meters$$

We decide to test our theory to see if this prediction is true. We find a higher bridge and set up a camera to take photos every second as the ball falls. Then we measure the distance from second to second.

4) Are you surprised by the results? Why or why not?



5) This is a photo of a basketball falling for 9 seconds. What do you notice? What questions do you have?



6) How far did the ball travel in the first second?

7) How far did the ball travel in the next second?

8) Why do you think the ball traveled so far in the 4th second?

9) Make predictions for the distance the ball falls after 8 and 9 seconds.

Were you able to predict the distance the ball falls after 8, 9, and 10 seconds? If so, you probably looked at the rate of change in the distance traveled:

- Between 0 and 1 second, the ball fell 5 meters.
- Between 1 and 2 seconds, the ball fell 15 meters.

10) Calculate the rate of change for the other time intervals in the table.

11) How far would the ball fall after 10 seconds?

When we look at the rate of change for this table, we can see that the difference between each output is changing as time passes. The change is -5, then -15, then -25, then -35, etc. The rate of change is changing! This tells us that the data is nonlinear.

12) In your opinion, why isn't the ball falling the same distance in each second? Why is the distance traveled changing?

time (seconds)	distance (meters)	rate of change
t	$d(t)$	
0	0	
		−5
1	-5	
2	-20	
3	-45	
4	-80	
5	-125	
6	-180	
7	-245	
8	-320	
9	-405	
10		

Think about what happens when we drop a ball and gravity pulls the ball towards the ground. In the instant the ball leaves our hand, it is barely moving, but as time passes, it goes faster and faster. This is *acceleration*. The speed of the ball increases as it falls through the air. In the first second, the average speed is -5 meters per second. In the next second, the average speed has increased to -15 meters per second.

Look at the table on the right. There is a change in the rate of change as each second goes by. This is the acceleration of the ball as gravity pulls it towards the ground.

Since there isn't a constant rate of change in the distance, there is not a linear relationship between time and distance. However, there is a pattern in the rate of change that can help us make predictions. What patterns do you see in how the average speed changes?

time (seconds)	distance (meters)	**average speed** (rate of change)	**acceleration** (change in the rate of change)
t	$d(t)$		
0	0		
1	-5	−5	
2	-20	−15	−10
3	-45	−25	
4	-80	−35	
5	-125	−45	
6	-180	−55	
7	-245	−65	
8	-320	−75	
9	-405	−85	
10	-500	−95	

13) Look at this equation for the difference between the next two rates of change: $-15 + \square = -25$. What is the value of the empty square? (This is the change between -15 and -25.)

14) Find the changes in the rate of change and add them to the diagram above.

When an object is dropped from a height, acceleration makes it go faster and faster until the moment it hits the ground[1]. The speed of the object increases as it falls. On Earth, gravity accelerates falling objects at a rate of -9.8 meters (-32 feet) per second per second. In our calculations so far, we have used an approximate acceleration of -10 meters per second per second so far. This means that every second a falling object will increase its speed by about 10 meters per second. The precise acceleration due to gravity is actually -9.8 m/s^2.

Look at the table on the right. These are the same measurements we examined in the introduction. You may have noticed that the rate of change (or average speed) itself changes 10 meters every second.

time (seconds)	distance (meters)
t	$d(t)$
0	0
1	-5
2	-20
3	-45
4	-80
5	-125
6	-180
7	-245
8	-320
9	-405
10	-500

average speed (rate of change)	**acceleration** (change in the rate of change)
-5	
	-10
-15	
	-10
-25	
	-10
-35	
	-10
-45	
	-10
-55	
	-10
-65	
	-10
-75	
	-10
-85	
	-10
-95	

Earlier, we thought the speed of the falling basketball might be -15 meters per second since this is the average rate of change from 0 to 3 seconds:

$$r = \frac{-45\ meters}{3\ seconds}$$
$$r = -15\ mps$$

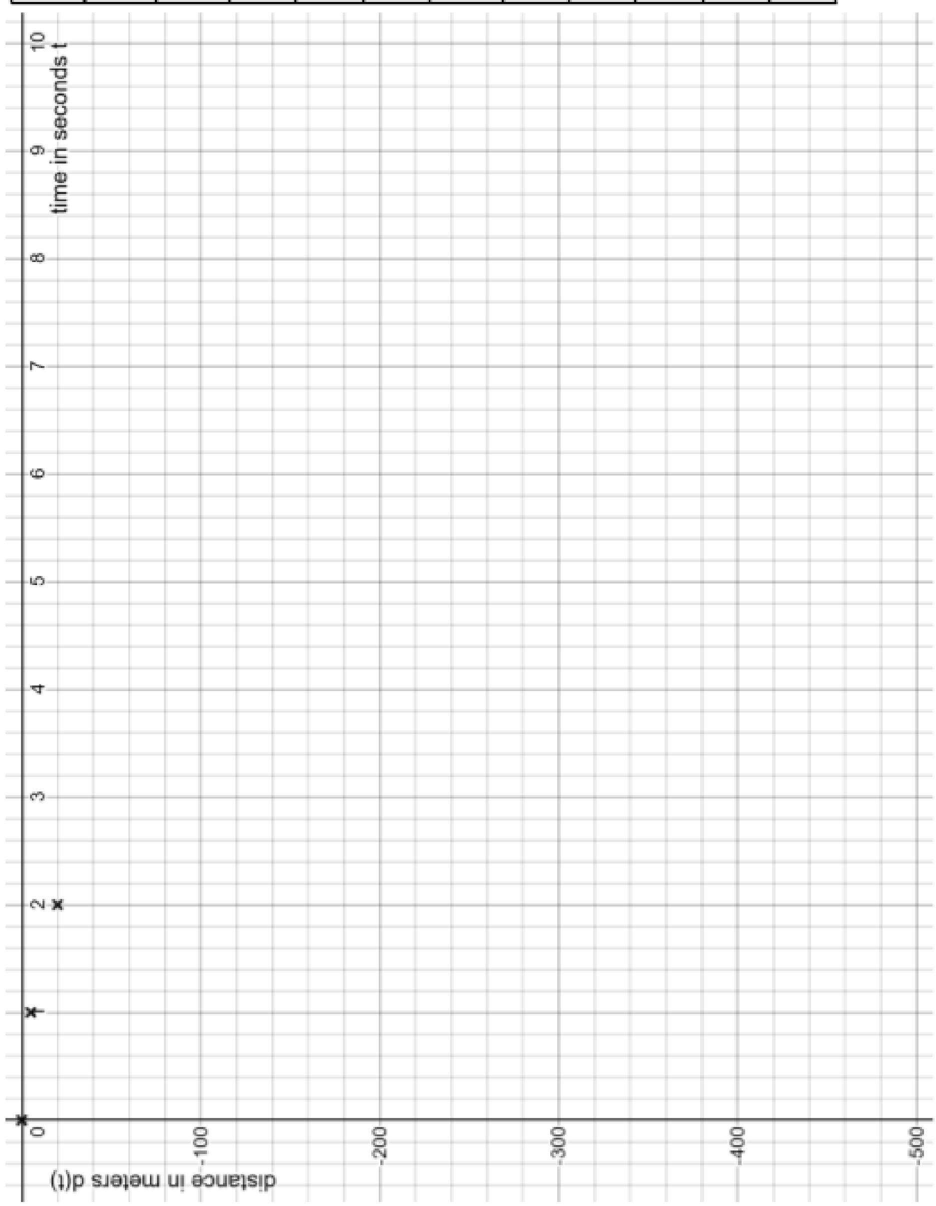
Since the speed is changing constantly, the average over the first 3 seconds can't be used to predict the distance traveled later, but it helps us understand how fast the ball has been falling so far.

15) What is the average rate of change in the distance traveled from 0 to 6 seconds?

[1] In the real world, friction in the air reduces acceleration and eventually causes *terminal velocity*, which is the maximum speed a falling object will reach. We are ignoring air resistance in this problem.

16) Create a graph of the data from the falling basketball test.

t	$d(t)$
0	0
1	-5
2	-20
3	-45
4	-80
5	-125
6	-180
7	-245
8	-320
9	-405
10	-500



This is a graph of the distance $d(t)$ the basketball has fallen after t seconds.

Draw x's for the missing points in the table. The first three points have been plotted for you.

Complete the two tables, then calculate the rate of change and change in the rate of change. This will help us find rules for these nonlinear functions. You may want to use a calculator.

17) Acceleration in feet per second^2

time (seconds)	distance (feet)
t	$d(t)$
0	0
1	-16
2	-64
3	-144
4	-256
5	-400
6	
7	
8	
9	
10	

−16

−48

Which function matches this table of data?

A. $d(t) = \frac{1}{2}(-32)t$

B. $d(t) = \frac{1}{2}(-32)t^2$

C. $d(t) = -16t$

D. $d(t) = -48t^2$

18) Acceleration in meters per second^2

time (seconds)	distance (meters)
t	$d(t)$
0	0
1	-4.9
2	-19.6
3	-44.1
4	-78.4
5	-122.5
6	
7	
8	
9	
10	

−4.9

−14.7

Which function matches this table of data?

A. $d(t) = \frac{1}{2}(-9.8)t$

B. $d(t) = \frac{1}{2}(-9.8)t^2$

C. $d(t) = -4.9t$

D. $d(t) = -14.7t^2$

Measuring acceleration

	in feet	in meters
Acceleration due to Earth's gravity is usually called "g". The direction of this acceleration is downward (toward the center of the Earth).[2]	$-32\ ft/s^2$ This means a falling object will accelerate downward, moving 32 feet per second faster every second.	$-9.8\ m/s^2$ This means a falling object will accelerate downward, moving 9.8 meters per second faster every second. (Earlier, we used 10 meters per second^2 as an approximate measurement.)
Both of these functions can be used to calculate the distance of a falling object over time. We can use either one, depending on whether we want the answer in feet or meters.	$f(x) = \frac{1}{2}(-32)x^2$ *The function of* x refers to the distance in feet traveled by a falling object. The variable x is the amount of time in seconds.	$f(x) = \frac{1}{2}(-9.8)x^2$ *The function of* x *r*efers to the distance in meters traveled by a falling object. The variable x is the amount of time in seconds.

19) How far would the ball fall in 15 seconds? Calculate your answer in feet and then in meters.

    Feet:

    Meters:

20) The equation $1936 = \frac{1}{2}(-32)x^2$ describes a ball falling 1,936 feet. How many seconds was it in the air?

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[2] The acceleration due to Earth's gravity is normally calculated using -9.8 m/s^2, the measurement in meters per second squared. The strength of gravity's force is due to Earth's size and is different on other planets. Smaller planets like Mercury have a lower acceleration (-3.7 m/s^2) and larger planets like Jupiter have a larger acceleration (-25 m/s^2) due to the differences in the force of gravity on those planets.

Why 9.8 and 32?

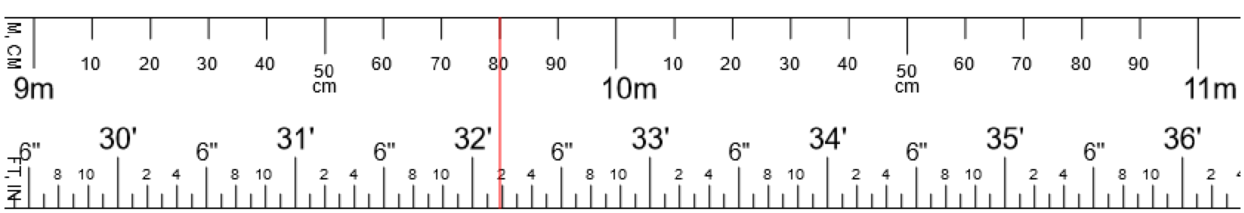
The measurement $-9.8\ m/s^2$ is the standard used by scientists around the world, including the United States. We often use $-32\ ft/s^2$ as well, since most of our calculations for distance are in feet and miles, rather than meters and kilometers. There are about 3.28 feet in a meter.

21) Calculate the missing conversions.

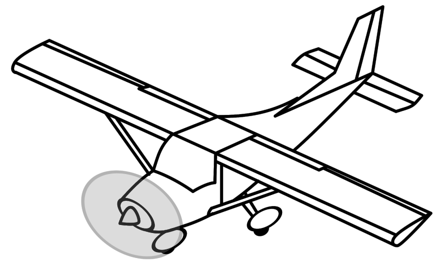
meter	Convert meters to feet	feet
1	1×3.28	3.28
	2×3.28	6.56
6		
10		
		16.4
.5	$.5 \times 3.28$	
7.5		
9		
9.8		

When you complete the last line of the table above, you should see that 32 feet and 9.8 meters are about the same distance.

22) Is 9.8 meters longer or shorter than 32 feet? By how much?



If you line up the two measurements on a meter stick (shown on the top) and a yard stick (shown on the bottom), you will see that 9.8 meters lines up close to 32 feet, but not exactly. 9.8 meters is equal to 9 meters and 80 centimeters, which is equivalent to 32 feet and almost 2 inches.



A ball is dropped out of an airplane and hits the ground 25 seconds later.

The function $f(t) = \frac{1}{2}(-32)t^2$ can be used to calculate how far an object falls, measured in number of feet. The variable t stands for time in seconds. $f(t)$ is the distance in feet that the object has fallen after t seconds. 32 ft/s^2 is the acceleration from gravity measured in feet.

23) Ignoring the effects of air resistance and wind, use the function to determine how far the plane is above the ground.

Use the table if it is helpful.

t	½ · (-32) · t²	f(t)
1	½ · (–32) · (1)²	–16
2	½ · (–32) · (2)²	–64

Shooting a Water Balloon

A water balloon is shot from a launcher. The balloon's height h in feet after t seconds can be calculated with the function $h(t) = 64t - 16t^2$. We can use the table below to calculate the height of the water balloon after it is shot from the launcher.

24) Complete the table.

25) $h(0)$ is 0 feet and $h(.5)$ is 28 feet. What is $h(1)$?

26) Why are some of the outputs equal?

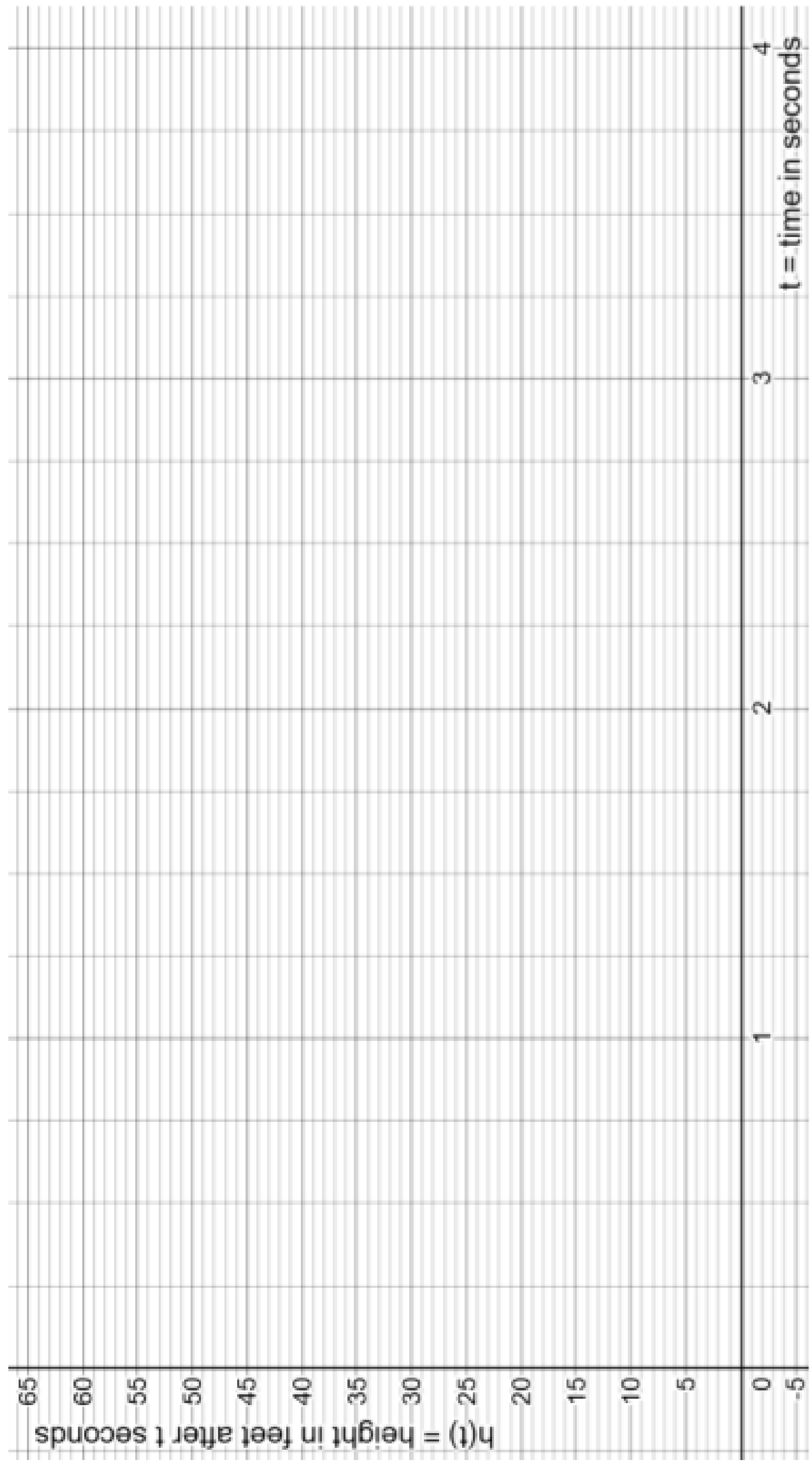
27) What is the highest point the water balloon reaches?

28) How long does the water balloon stay in the air?

29) Is there anything strange about $h(4.5)$?

time (seconds)	function calculation	height (feet)
t	$h(t) = 64t - 16t^2$	$h(t)$
0	$64 \times 0 - 16 \times 0^2$ $0 - 0$	0
.5	$64 \times .5 - 16 \times .5^2$ $32 - 4$	
1		
1.5		
2		
2.5		
3		
3.5		
4		
4.5		

30) Make a graph of the height $h(t)$ of the water balloon after t seconds. Use the data from the previous page.

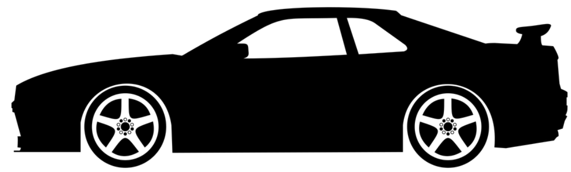


31) What do you notice when you look at the graph of the data points?

32) There should be 4 pairs of equal outputs. For example, the height after 0 seconds should be the same as the height after 4 seconds. What are the other three pairs of outputs?

33) Why do we not include $h(4.5)$ in the graph?

Acceleration in a Car



Carthian bought a used car recently. She did some research on the Internet and found data from a road test with the same kind of car.

The data shows the results of a 5-second test, in which the driver started from a stationary position and accelerated (increased speed) until the time was up.

Please do not try this yourself. You can look up the acceleration times for many vehicles online.

Road test for the Elektro Rapida

Time (seconds)	Distance (feet)
0	0.0
1	5.5
2	22.0
3	49.5
4	88.0
5	137.5

34) How far did the car travel in the first second?

35) How far did the car travel in the next second (between 1 and 2 seconds)?

36) What patterns do you notice in the rate of change?

37) What was the average rate of change[3] during the complete road test? Explain how you know.

[3] The average rate of change is the same as the average speed over the time full timed test.

38) How far would Elektro Rapida travel in 10 seconds if it continued to accelerate at the same rate?

A. 176 feet

B. 275 feet

C. 385 feet

D. 550 feet

39) The formula for distance traveled at a constantly accelerating speed is: $d(t) = (\frac{1}{2}) \cdot a \cdot t^2$ where a stands for *acceleration* and t stands for time. What is the correct value of a for the Elektro Rapida?

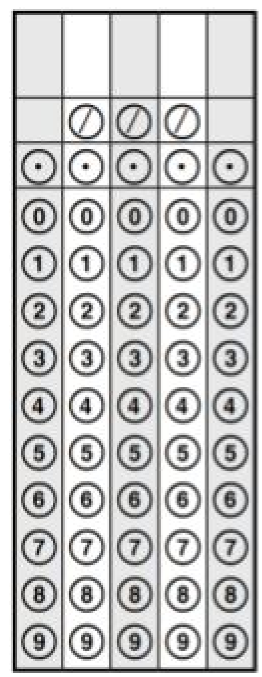
A. 5.5 feet per second^2

B. 11 feet per second^2

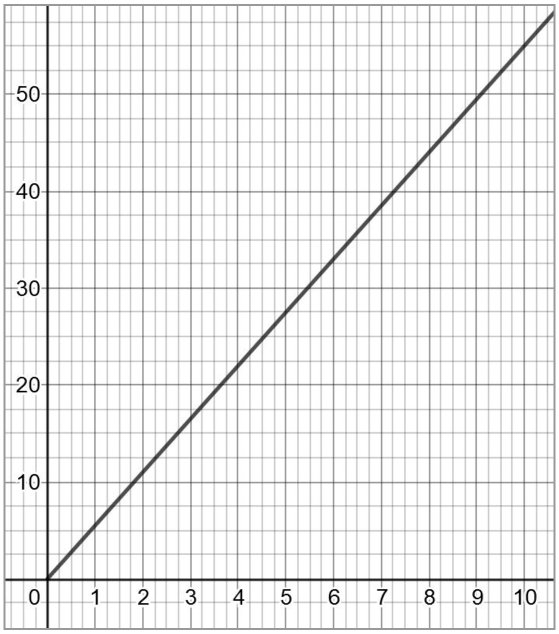
C. 16.5 feet per second^2

D. 55 feet per second^2

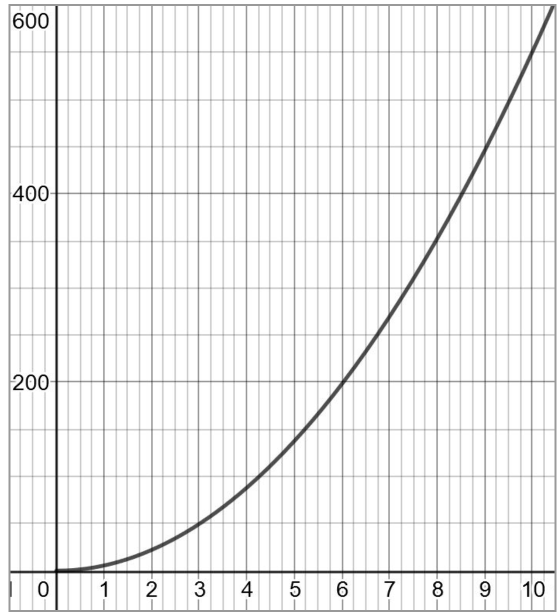
40) It takes the Elektro Rapida 8 seconds to go from 0-60 mph. How many feet does it travel in 8 seconds? Write your answer in the grid to the right.



41) Which graph matches the time and distance data for the road test of the Elektro Rapida?



A.



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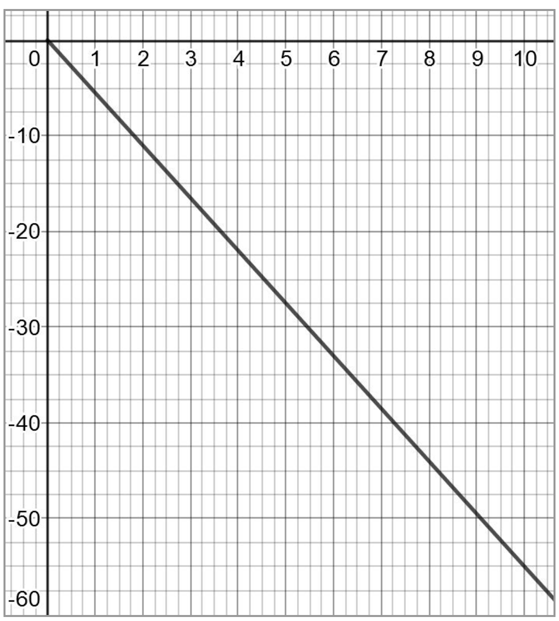
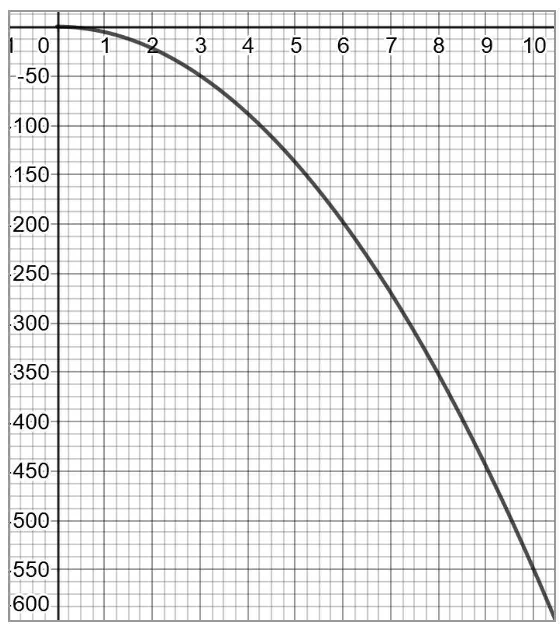


C.



D.

Braking Distance

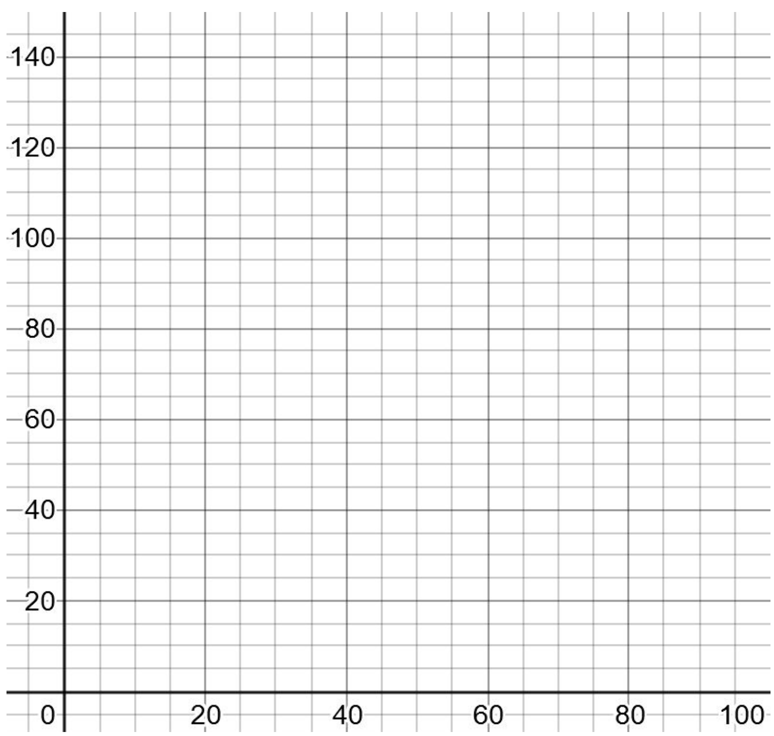
When we talk about speed in the United States, we usually use miles per hour. In this section it will be useful to give speed in feet per second.

42) 100 miles per hour (mph) is about 147 feet per second (fps). How many fps is 50 mph?

43) Complete the table to show the relationship of miles per hour to feet per second. Extend the table up to 80 miles per hour.

mph	fps
10	14.7
20	
30	

44) If you made a graph with mph on the x-axis and fps on the y-axis, what would the graph look like?



45) If you were traveling at 1 mph, how fast would you be going in fps?

The speed of a car matters in a crash. A crash at higher speed contains more energy which causes more damage to you, your vehicle, and possibly other people. Speed also matters when you need to brake. Your own reaction time (how quickly you see a threat and then decide what action to take) takes place before you put your foot on the brake. Once you hit the brake, it takes time for the car to slow down.

To stop a car in an emergency, you first react and then put on the brakes.

$$\textit{stopping distance} = \textit{reaction distance} + \textit{braking distance}$$

46) What kinds of things do you think would affect reaction time and distance traveled before hitting the brakes?

47) What kinds of things do you think would affect braking time and distance?

Reaction distance

Reaction time is considered to be about 1.5 seconds on average, but how far you travel during this time depends on how fast you are going. This is how far your car travels **before** you hit the brakes, while you are reacting to an emergency.

48) Complete the table to see how a car travels in 1.5 seconds at different speeds.

Speed (mph)	Speed (fps)	Calculation	Reaction Distance (feet)
10	14.7	14.7×1.5	22.05
20	29.4	29.4×1.5	
30	44.1		
40			
50			

Braking distance

After 1.5 seconds of reaction time, you hit the brakes. Your car continues to move, but it slows down. The distance that your car travels after braking depends on the vehicle's brakes, the conditions of the road, the weight of your vehicle, and the condition of your tires. Mathematicians who work in government planning and road construction have formulas to estimate how much distance cars need to slow down at different speeds.

A rule for finding braking distance in feet is to take the speed in miles per hour, square it, and divide the result by 20. For example, if the speed were 10 mph, the braking distance would be $10^2 \div 20 = 100/20 = 5\,feet$. This is how far a car travels **after** hitting the brakes.

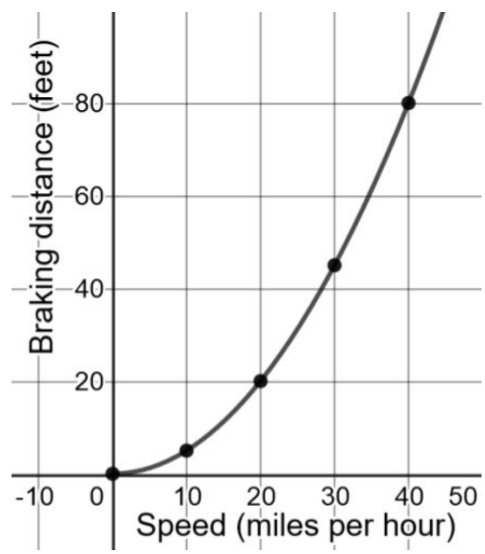
49) How would you write the braking distance formula as a function?

A. $f(x) = \frac{x^2}{20}$

B. $f(x) = \frac{20}{x^2}$

C. $f(x) = 20x^2$

D. $f(x) = (\frac{x}{20})^2$



The graph shows the relationship between speed (in miles per hour) and braking distance (in feet). All the points on the graph were found using the formula above.

50) Make a table showing the coordinates of at least five points on the graph.

Speed (mph)	Calculation	Braking Distance (feet)
10	$10^2 \div 20$	5

51) Is braking distance an example of linear or nonlinear growth? Why?

52) If you double your speed, will you double your braking distance? Explain, giving examples.

Total stopping distance

53) Use your data from **reaction distance** and **braking distance** to create the table below. Use at least five different speeds.

Speed (mph)	Reaction Distance (feet)	Braking Distance (feet)	Total Stopping Distance (feet)
10	22.05	5	27.05
20	44.1	20	64.1
30	66.15	45	
40	88.2	80	
50	110.25		
60			
70			
80			
90			

Safe distance between cars

Four of every 10 crashes involve rear-end collisions, normally because a person is following too closely (tailgating). In order to prevent crashes, it is important to know your safe following distance. Since calculating stopping distance is complicated, different ways of estimating safe following distance are sometimes recommended.

Rule 1: The 3-Second Rule. To calculate a safe following distance, notice when the vehicle in front of you passes some object, such as a road sign. Then time three seconds by counting, "One-thousand-one, one-thousand-two, one-thousand-three." If you pass the same object before you get to one-thousand-three, you are following too closely.

Rule 2: The 1-for-10 Rule. To calculate your following distance, leave one car length between you and the car in front of you for every 10 mph of driving speed. A safe following distance is the distance in car lengths.

54) Using what you have learned about the stopping distance of cars at different speeds, which rule do you think is safer? Why?

Helpful tips: Help the driver behind you by maintaining a safe following distance and a steady speed. Tap your brakes to warn the driver behind you when you plan to slow down or stop.

Increase your following distance when driving:

- in bad weather
- behind a large vehicle that blocks your vision
- in heavy traffic
- behind a motorcycle or bicycle

If a driver follows you too closely (tailgates), move to another lane if possible or reduce speed and pull off the road to let the driver go by. Make sure to signal when you drive off the road and when you return to it. Do not press your brakes suddenly or unnecessarily as this may startle the motorist behind you.

For more information on safe driving, visit https://dmv.ny.gov.

Making a Profit

The Aparato Company tried to sell a can opener for $24, but they weren't able to sell any, so they decided to try to attract customers by reducing their prices. They found that for every $1 they lowered the price, they sold 10 more can openers.

55) Complete the table.

Price Reduction (dollars)	Price (dollars)	Number of Can Openers Sold	Calculation	Gross Profit[4] (dollars)
0	24	0	$24 \cdot 0$	0
1	23	10	$23 \cdot 10$	230
2	22	20	$22 \cdot 20$	440
3				
4				
5				
6				
7				
8				
9				
10				
11				
12				
13				
14				

[4] *Gross profit* is the amount of money that a company makes before paying any expenses. A company making can openers might have expenses such as raw materials (steel and plastic), pay for employees, rent, insurance, and taxes. *Net profit* is the amount of money a company makes after subtracting expenses. In this problem, we are only looking at gross profit.

56) Analyze your table. What do you notice?

57) If the price is reduced by $5, how many people will buy a can opener?

58) If the price is reduced by $5, what would the gross profit be?

59) If Aparato keeps reducing the price of the can opener, will they make more and more money? Why or why not?

60) Based on the table, what would you advise Aparato to do with the price of the can opener?

61) If the price is reduced by x dollars, what is the new price of the can opener?

A. x

B. $x - 24$

C. $24 - x$

D. $24x$

62) If the price is reduced by x dollars, how many people will buy a can opener?

A. x

B. $10x$

C. $\frac{10}{x}$

D. $\frac{x}{10}$

63) Which function equation calculates the gross profit of the can opener sale?

A. $y = (x - 24)(10x)$

B. $y = (x - 10)(24x)$

C. $y = (10 - x)(24x)$

D. $y = (24 - x)(10x)$

64) The Aparato Company was trying to sell an item for *P* dollars, and no one was buying it. They found that for every \$1 they lower the price (x), they gain *C* customers. Which function below could be used to analyze this situation?

A. $Gross\ profit = (P - x)(Cx)$

B. $Gross\ profit = (x - P)(Cx)$

C. $Gross\ profit = (x - C)(Px)$

D. $Gross\ profit = (C - x)(Px)$

Introduction to Exponential Functions

Number Patterns and Exponential Growth

In part 1 of this packet, we examined a mix of linear, quadratic, and exponential growth patterns to see if we could predict the next few numbers in each series. There is one linear pattern and one quadratic pattern below.

	Number Pattern	Description of Pattern	Function Equation
Linear function:	7, 14, 21, 28, 35, …	Add 7	$y = 7x$
Quadratic function:	2, 6, 12, 20, 30, …	Add 4, 6, 8, 10, etc.	$y = x^2 + x$

Make sure you understand how each function equation produces the number patterns above.

1) Complete the two function tables.

Equation: $y = 7x$							
x	1	2	3	4	5	6	7
y							

Equation: $y = x^2 + x$							
x	1	2	3	4	5	6	7
y							

The functions below are exponential. Their pattern is different from linear and quadratic functions.

Continue the pattern. Describe the pattern.

2) 1, 2, 4, 8, 16, 32, _____ , _____ , ...

3) 1, 3, 9, 27, 81, 243, _____ , _____ , ...

4) 16, 8, 4, 2, 1, ½, _____ , _____ , ...

In this section, we will analyze exponential functions so that we understand their pattern of growth, how to represent them in tables, equations, and graphs, and finally, how to use them to solve problems in the world. Exponential growth is a good tool for understanding change related to science, finance, and technology. Understanding how to use exponential growth will give you the power to understand scientific and financial situations.

In the high school equivalency test, you may be given a number pattern, an equation, a table, a graph, or a visual pattern. You will need to decide whether you are looking at a linear, quadratic, or exponential function. From previous practice, we know how to tell the difference between linear and nonlinear functions. In this section, we will practice looking at exponential functions so that you recognize how they are different from linear and quadratic functions.

Understanding how numbers grow or decay[5] in number patterns is a good place to start. Let's look at the three exponential number patterns above:

Number Pattern		Description of Pattern
1, 2, 4, 8, 16, 32, 64, 128, ...	→	Multiply by 2
1, 3, 9, 27, 81, 243, 729, 2187, ...	→	Multiply by 3
16, 8, 4, 2, 1, ½, ¼, ⅛, ...	→	Multiply by ½ (or divide by 2)

5) What do you notice when you compare the pattern descriptions above with the descriptions of the linear and quadratic functions above?

   Linear: Add 7 Quadratic: Add 4, 6, 8, 10, etc.

In a linear function, the same value is added to each previous number. In a quadratic function, we add a changing series of numbers each time to get the next number in the pattern. In exponential functions, we multiply to get the next number in the series!

6) Fill in the next two numbers in this exponential number series:

1	2	4	8	16	32	64	128		

× 2 × 2 × 2 × 2 × 2 × 2 × 2 × 2 × 2

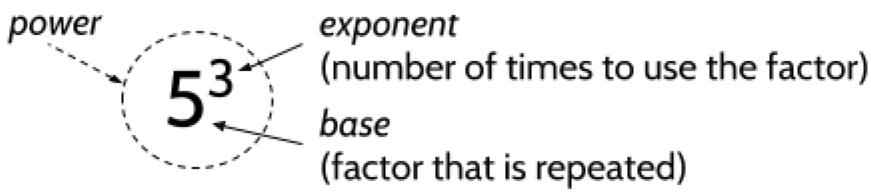
[5] get smaller

Exponential Growth is Repeated Multiplication

In **The Power of Exponents**, we learned how to calculate the value of numbers in exponential form, simplify calculations with very large and very small numbers, and use exponents to solve problems. As we learned in that packet, exponents are a way to show repeated multiplication.

Let's review a few basics of exponents.

The number 5^3 is an example of a number written as a *power*. 5^3 means $5 \times 5 \times 5$ or 125. When a number is written as a power, it has a *base* and an *exponent*. The exponent (the little number on top) tells us how many times the base (the number on the bottom) should be used as a factor in multiplication. This is also called *exponential form*.



In this number, the 5 should be used in multiplication 3 times: $5 \times 5 \times 5 = 125$.

There are many ways to read 5^3. The formal way to say it is:

five raised to the third power

It can also be shortened to *five to the third power* or *five to the third*. Another way to say the same thing is *five cubed*.

7) What is the value of 2^6 ?

8) What is the value of 4 raised to the 3rd power?

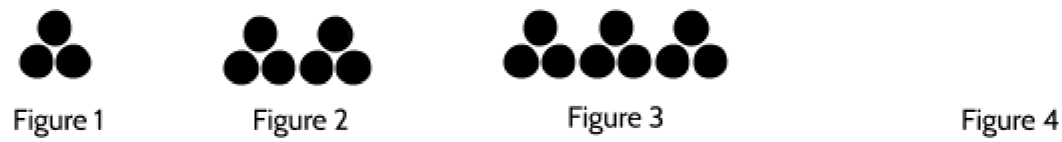
9) If $2^x = 16$, what value of x makes the equation true?

You can find more practice with exponents in **The Power of Exponents** study packet.

Visual Patterns

10) Look at Visual Pattern F, then follow the instructions below.

a) Draw Figure 4.



Visual Pattern F

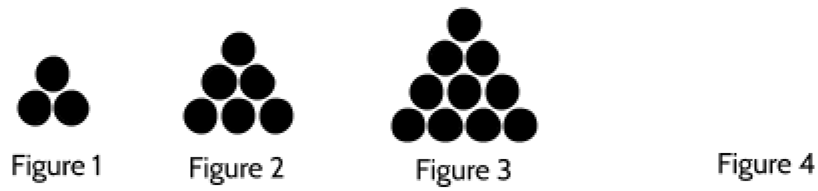
b) Complete the table.

Figure	Number of Circles
1	3
2	6
3	
4	
5	
6	

c) How is this visual pattern growing?

11) Look at Visual Pattern G, then follow the instructions below.

a) Draw Figure 4.



Visual Pattern G

b) Complete the table.

Figure	Number of Circles
1	3
2	6
3	
4	
5	
6	

c) How is this visual pattern growing?

12) Look at Visual Pattern H, then follow the instructions below.

a) Draw Figure 4.

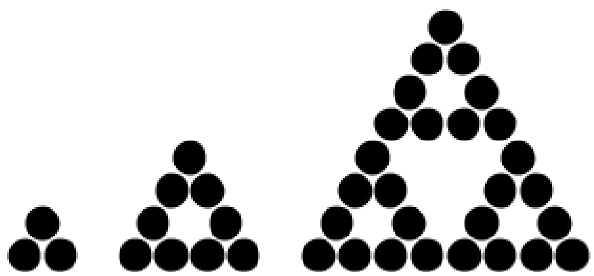


Figure 1 Figure 2 Figure 3 Figure 4

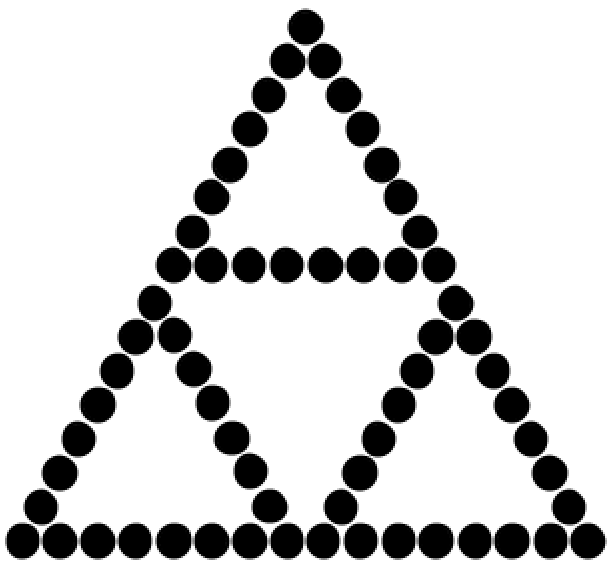
Visual Pattern H

b) Complete the table.

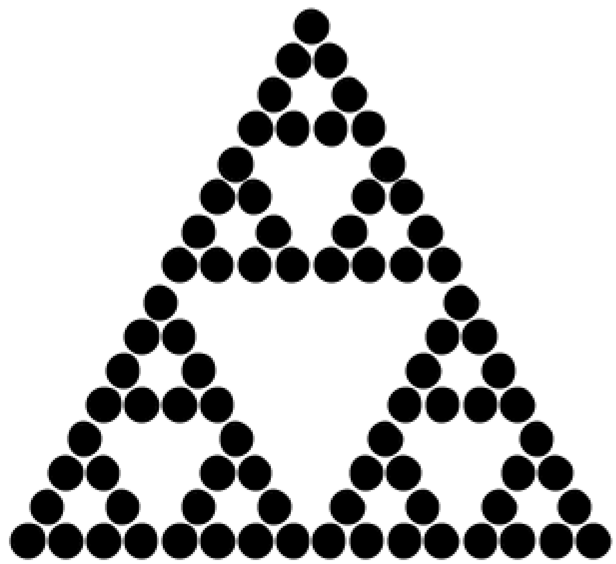
Figure	Number of Circles
1	3
2	9
3	
4	
5	
6	

c) How is this visual pattern growing?

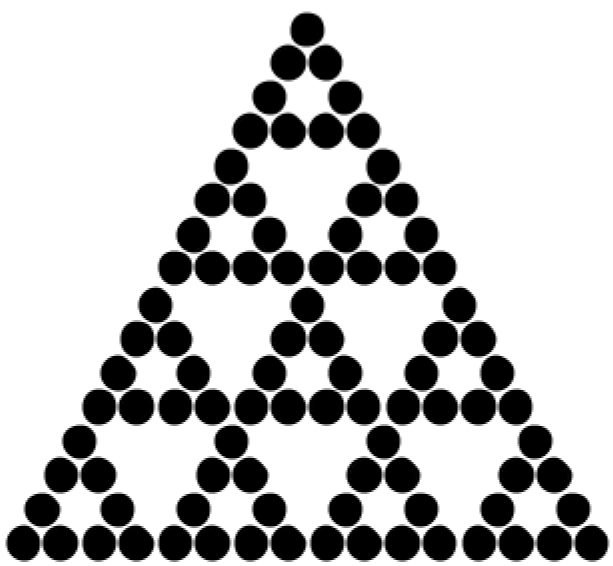
Does your drawing for Figure 4 of Visual Pattern H look like one of the figures below?



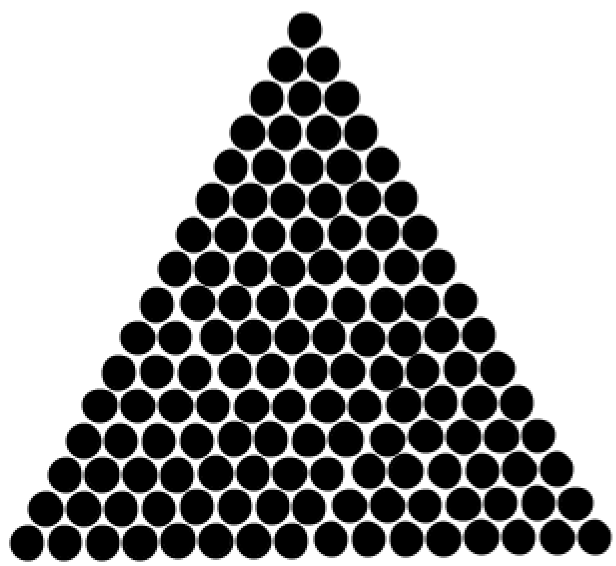
I



II



III

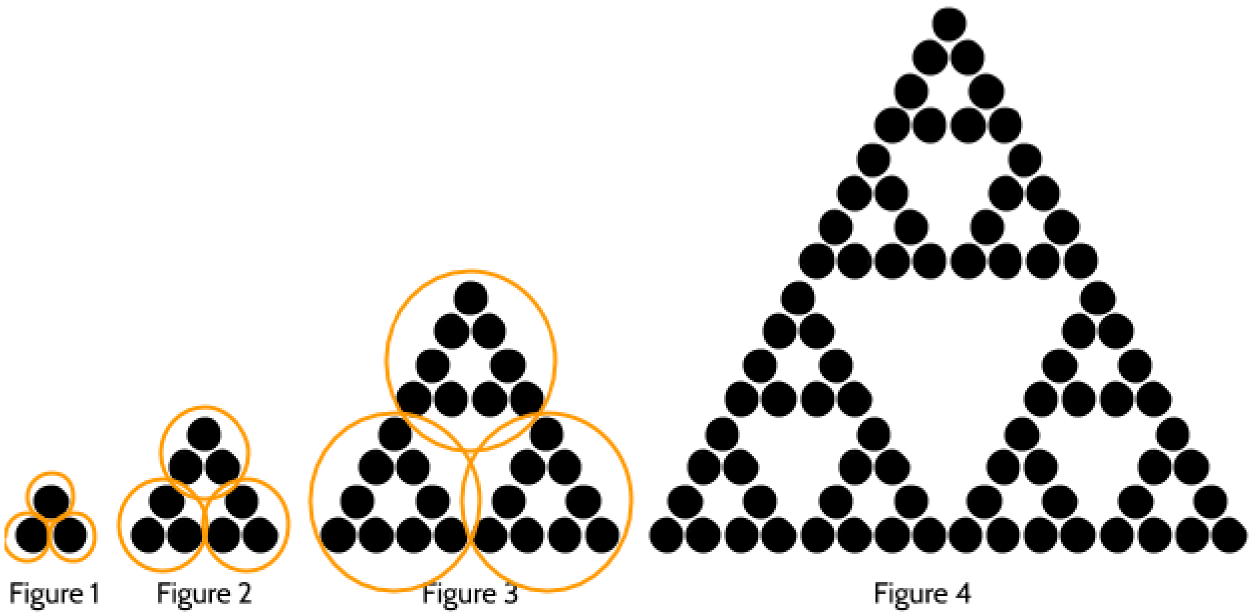


IV

According to the way we see the pattern growing, Figure 4 would look like Drawing II. (However, you might argue for different possibilities.)

13) How many circles are in Figure 4 (Drawing II)?

When we look at Visual Pattern H, we see each figure being multiplied by 3. For example, Figure 1 has three circles. Those three circles are repeated 3 times in Figure 2. Then Figure 2 is repeated 3 times in Figure 3.



Look at Figure 4. Can you see Figure 3 inside figure 4?

14) Draw outlines to split Figure 4 in three pieces.

15) Add the number of circles to the table for Visual Pattern H below. Can you use a pattern to fill in the missing outputs?

Fig. No.	1	2	3	4	5	6	7
No. of Circles	3	9	27				

Complete the following sentences.

16) Visual Pattern F is a ________________________ growth pattern because the rate of change is constant.

17) If the function equation for Visual Pattern G includes n^2, it must be a ________________________ growth pattern.

18) Visual Pattern H is an example of an exponential growth pattern because it grows by multiplying by a factor of _______ to create each new figure.

19) Visual Pattern G and Visual Pattern H are both ________________________ growth patterns because their rates of change are not constant.

Match the function equation with the function tables below.

20) $f(n) = 3^n$ → Visual Pattern _________

21) $f(n) = 3n$ → Visual Pattern _________

22) $f(n) = \frac{(n+1)(n+2)}{2}$ → Visual Pattern _________

Visual Pattern F

Figure	Number of Circles
1	3
2	6
3	9
4	12
5	15
6	18

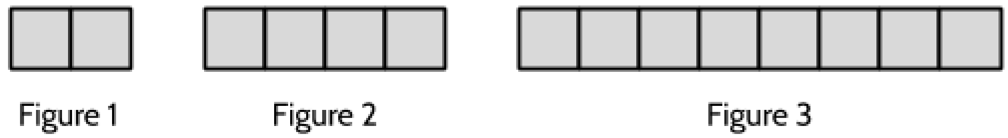
Visual Pattern G

Figure	Number of Circles
1	3
2	6
3	10
4	15
5	21
6	28

Visual Pattern H

Figure	Number of Circles
1	3
2	9
3	27
4	81
5	243
6	729

23) Consider the pattern of squares shown below:



Which type of function, linear, quadratic, or exponential, should be used to find the number of squares in the 99th figure? Explain your answer.

Three Views of an Exponential Function

We have now used looked at linear functions and quadratic functions with the three views of a function: a table, an equation, and a graph. Exponential functions can also be represented with each of these views. We will see differences when we compare exponential tables, equations, and graphs with representations of linear and quadratic functions.

Tables

Let's start by looking at some functions in a table format. Use number patterns to find the missing outputs.

1) Complete the table.

In	*Out*
0	0
1	2
2	4
3	
4	8
5	

How did you find the missing *Out* numbers?

2) Complete the table.

In	*Out*
0	1
1	2
2	4
3	8
4	
5	

How did you find the missing *Out* numbers?

3) What difference do you see between these two functions?

In	*Out*	rate of change
0	0	
1	2	+ 2
2	4	+ 2
3	6	+ 2
4	8	+ 2
5	10	+ 2

In	*Out*	growth factor
0	1	
1	2	× 2
2	4	× 2
3	8	× 2
4	16	× 2
5	32	× 2

The first table on the previous page is for a linear function. We can tell that is a linear function because the *rate of change* is constant, with 2 added between each output.

The second table on the previous page is for an exponential function. The outputs from the function increase through a *growth factor* rather than a rate of change. Instead of adding 2 to each previous output, we multiply each previous output by 2.

4) Complete the outputs in these two function tables, then write the growth factor.

In	*Out*	
0	1	
1	4	× 4
2	16	
3	64	
4		
5		

In	*Out*
0	1
1	5
2	25
3	125
4	
5	

If you know the growth factor in an exponential growth pattern, you can move backwards and forwards to figure out the outputs of a table.

5) Fill in the missing outputs.

In	0	1	2	3	4	5
Out		10	20	40		

$\times 2$ $\quad\times 2$ $\quad\times 2$ $\quad\times 2$ $\quad\times 2$

Since the growth factor in the function above is 2, each number is multiplied by 2 to get the next number. Going from left to right in the table above:

$5 \cdot 2 = 10 \qquad 10 \cdot 2 = 20 \qquad 20 \cdot 2 = 40 \qquad 40 \cdot 2 = 80 \qquad 80 \cdot 2 = 160$

This means that each number in the series can be divided by 2 to get the number before it. Starting on the right and moving left:

$160 \div 2 = 80 \qquad 80 \div 2 = 40 \qquad 40 \div 2 = 20 \qquad 20 \div 2 = 10 \qquad 10 \div 2 = 5$

What happens if you divide each number by the number before it?

6) Complete the following equations.

$160 \div 80 = \qquad 80 \div 40 = \qquad 40 \div 20 = \qquad 20 \div 10 = \qquad 10 \div 5 =$

Did you get the same answer each time? This is a property[1] of exponential number patterns. If you divide each number by the number before it in the pattern, you should get the same number. This number is the growth factor.

You can use this property to test if a number pattern is exponential. If you divide consecutive outputs in the pattern and get the same number as an answer each time, the number series is exponential.

7) Is this number series exponential?

In	1	2	3	3	4	5
Out	6	18	54	162	486	1458

[1] A character or quality that something has.

8) You can check to see if the previous function is exponential by dividing each number by the number before it in the sequence. Find the quotient[2] of each equation below.

$1458 \div 486 =$ $486 \div 162 =$ $162 \div 54 =$ $54 \div 18 =$ $18 \div 6 =$

The function above is exponential because 3 is the growth factor between each output. You can multiply by 3 to get the next number to the right and you can divide by 3 to get the next number to the left.

For each of the following functions:

- Decide if there is a constant growth factor.
- Decide if the function is exponential.

9)

x	0	1	2	3	4
y	3	12	48	192	768

10)

x	0	1	2	3	4
y	2	4	12	36	72

11)

x	0	1	2	3	4
y	3	4.5	6.75	10.125	15.1875

12)

x	0	1	2	3	4
y	4096	2048	1024	512	256

[2] Result of dividing two numbers.

Sometimes, the growth factor is not a whole number. In this function, the growth factor is 1.5. You can multiply each number by 1.5 to get the next number.

In	0	1	2	3	4
Out	3	4.5	6.75	10.125	15.1875

× 1.5 × 1.5 × 1.5 × 1.5

Other times, the factor is less than 1. In this function, each number is multiplied by .5 to get the next number:

In	0	1	2	3	4
Out	4096	2048	1024	512	256

× .5 × .5 × .5 × .5

In this function, the outputs are getting smaller as the inputs get larger. Each time we multiply by .5, the number is half as big as it was. This is an example of an exponential *decay* function. In math, *decay* means "to go down." We might call .5 a *decay factor* instead of a growth factor since the numbers are getting smaller instead of growing. In each step in the pattern, the outputs decay by a factor of .5.

How do you know if the numbers will grow or decay? The next two functions have the same starting number, but they have a different factor that produces the next number in each series. What happens to the outputs based on the different factors?

13) Fill in the missing outputs.

Factor → 1.1

Is it growth or decay?

In	0	1	2	3	4
Out	10				

× 1.1 × 1.1 × 1.1 × 1.1

Factor → .9

Is it growth or decay?

In	0	1	2	3	4
Out	10				

× .9 × .9 × .9 × .9

14) Would a factor of .99 make numbers grow or decay?

15) Would a factor of 1.01 make numbers grow or decay?

Equations

Exponential functions have a growth pattern based on multiplying by the same factor over and over. For example, in the pattern below, each number is multiplied by a factor of 5 to get the next number.

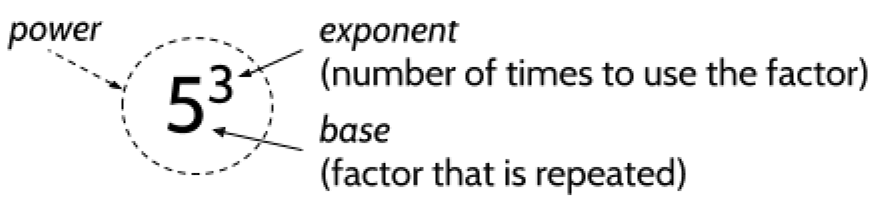
16) Write in the next two numbers in the series. Feel free to use a calculator.

In	0	1	2	3	4	5
Out	1	5	25	125		

$\times 5$ $\times 5$ $\times 5$ $\times 5$ $\times 5$

In an exponential function, the rule for the pattern above could be written as $Out = 5^{In}$ or $y = 5^x$. To get the output of the function, we use 5 as the base of a power. The input tells us the exponent to use with that base.

As we saw earlier, the base is the factor that is multiplied and the exponent tells us how many times to use the factor in multiplication.



To calculate the value of 5^3, we can expand it to $5 \cdot 5 \cdot 5$, which means the same thing. So, the value of 5^3 is 125.

17) If $y = 5^x$ and $x = 4$, what is the value of y?

18) If $y = 5^x$ and $x = 6$, what is the value of y?

19) If $y = 5^x$ and $y = 25$, what value of x makes the equation true?

20) If $y = 3^x$ and $x = 5$, what is the value of y?

21) In the table below, the number 1 is included as the starting amount of the function. The exponential form of each number and the factors of multiplication are included. Complete the table based on the equation.

Equation: $y = 3^x$			
x	Exponential Form	Factors	y
0		1	1
1	3^1	$1 \cdot 3$	3
2	3^2		9
3		$1 \cdot 3 \cdot 3 \cdot 3$	
4	3^4		

22) Complete the table based on the equation. Be careful: The inputs are out of order.

Equation: $y = (2.5)^x$			
x	Exponential Form	Factors	y
4		$1 \cdot 2.5 \cdot 2.5 \cdot 2.5 \cdot 2.5$	
	$(2.5)^2$		6.25
0		1	1
6	$(2.5)^6$		
			15.625

In a quadratic function like x^2, the inputs change the base of the power x^2. This changes the factor being multiplied. If x is 5, then x^2 is $5 \cdot 5$. If x is 7, then x^2 is $7 \cdot 7$. The base is always used as a factor twice, since x^2 means $x \cdot x$.

In an exponential function like 2^x, in comparison, the inputs change the exponent of the power 2^x. This changes the number of times 2, the base, is multiplied. The variable x tells us how many times to use 2 as a factor. For example, if x is 5, then 2^x is $2 \cdot 2 \cdot 2 \cdot 2 \cdot 2$.

In the exponential function $y = 1 \cdot 2^x$, the number 1 is used as a starting amount.

Equation: $y = 1 \cdot 2^x$			
x	Exponential Form	Factors	y
0	$1 \cdot 2^0$	1	1
1	$1 \cdot 2^1$	$1 \cdot 2$	2
2	$1 \cdot 2^2$	$1 \cdot 2 \cdot 2$	4
3	$1 \cdot 2^2$	$1 \cdot 2 \cdot 2 \cdot 2$	8

What happens if we use other numbers as starting amounts?

23) Complete the table.

Equation: $y = 10 \cdot 2^x$			
x	Exponential Form	Factors	y
0	$10 \cdot 2^0$	10	10
1	$10 \cdot 2^1$	$10 \cdot 2$	
2	$10 \cdot 2^2$	$10 \cdot 2 \cdot 2$	
3			

24) Complete the table.

Equation: $y = 100 \cdot 2^x$			
x	Exponential Form	Factors	y
0	$100 \cdot 2^0$	100	
1	$100 \cdot 2^1$	$100 \cdot 2$	
2	$100 \cdot 2^2$		
3			

25) What do you notice?

Practice using exponential function equations to answer the following questions.

26) If $y = 50 \cdot 3^x$ and $x = 2$, what is y?

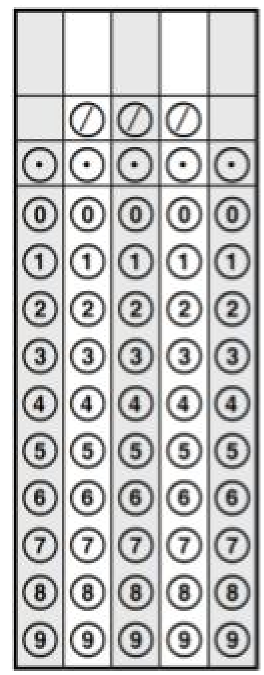
27) If $Out = 200 \cdot (.5)^{In}$ and $In = 4$, what value of Out makes the equation true?

28) To calculate the amount of yearly interest on a $2,000 investment making 2% interest, you can use the function:

$A = \$2,000 \cdot 1.02^t$

Where A is the total amount of the investment after t years that the investment has been earning interest.

If t is 5 years, what is the value of A? Round to the nearest dollar and enter into the grid on the right.



29) The function $P(t) = 1,000 \cdot 1.4^t$ models a population of deer in a state park that is measured over the course of a few years. 1,000 is the number of deer when the population was first counted. $P(t)$ is the size of the population after t years.

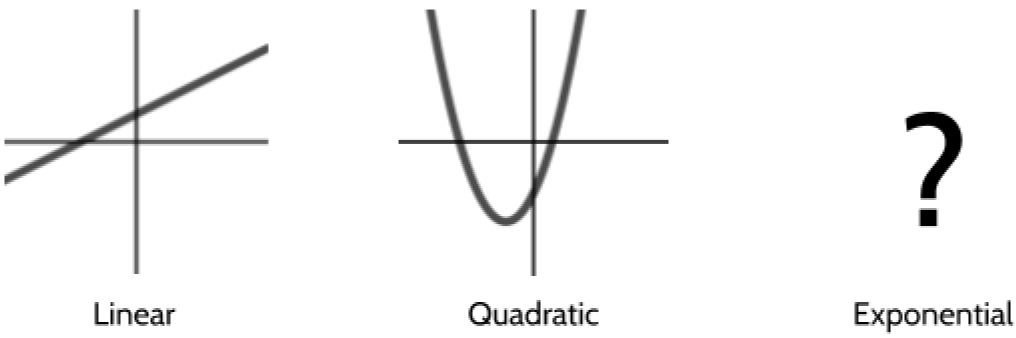
How many deer would you expect after 10 years? Round your answer to a whole number.

30) Is the function $P(t) = 1,000 \cdot 1.4^t$ an example of exponential growth or decay?

Graphs

By now, you have probably guessed that exponential functions are a kind of nonlinear growth. Instead of the constant rate of change we see in linear functions, there is a growth or decay factor. Instead of adding growing values to the previous number in the pattern, as we do in quadratic functions, we multiply by the same number over and over.

We can use graphs to see what exponential growth and understand how it is different from linear and quadratic growth. The graph of a linear function is a line and the graph of a quadratic function is a parabola. What does the graph of an exponential function look like?



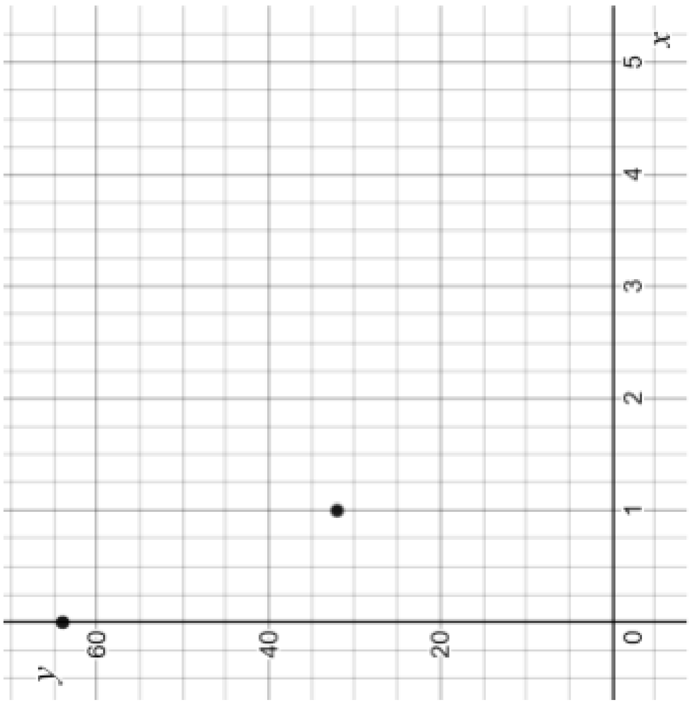
To find out, let's create a couple function tables and draw graphs.

31) Complete the tables.

Equation: $y = 1 \cdot 2^x$	
x	y
0	1
1	
2	
3	
4	
5	

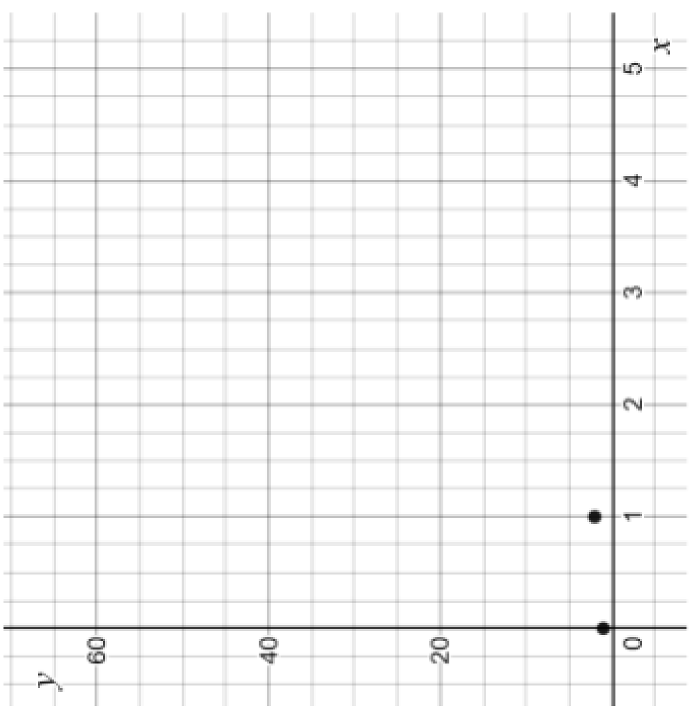
Equation: $y = 64 \cdot (.5)^x$	
x	y
0	64
1	
2	
3	
4	
5	

32) Create graphs from the function tables on the previous page.



Plot the data points for the function $\mathbf{y = 64 \cdot (.5)^x}$.

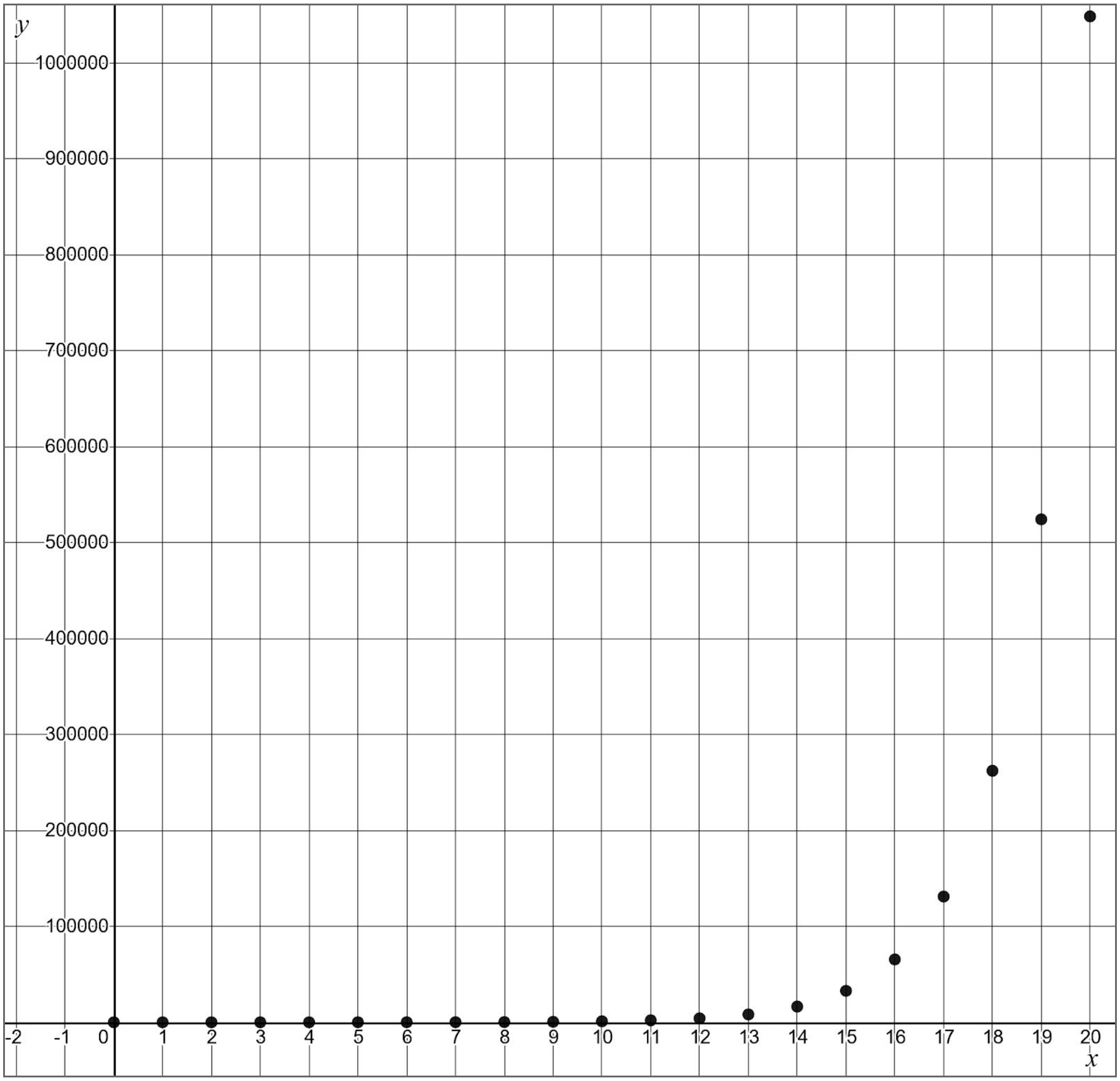
What do you notice?



Plot the data points for the function $\mathbf{y = 1 \cdot 2^x}$.

What do you notice?

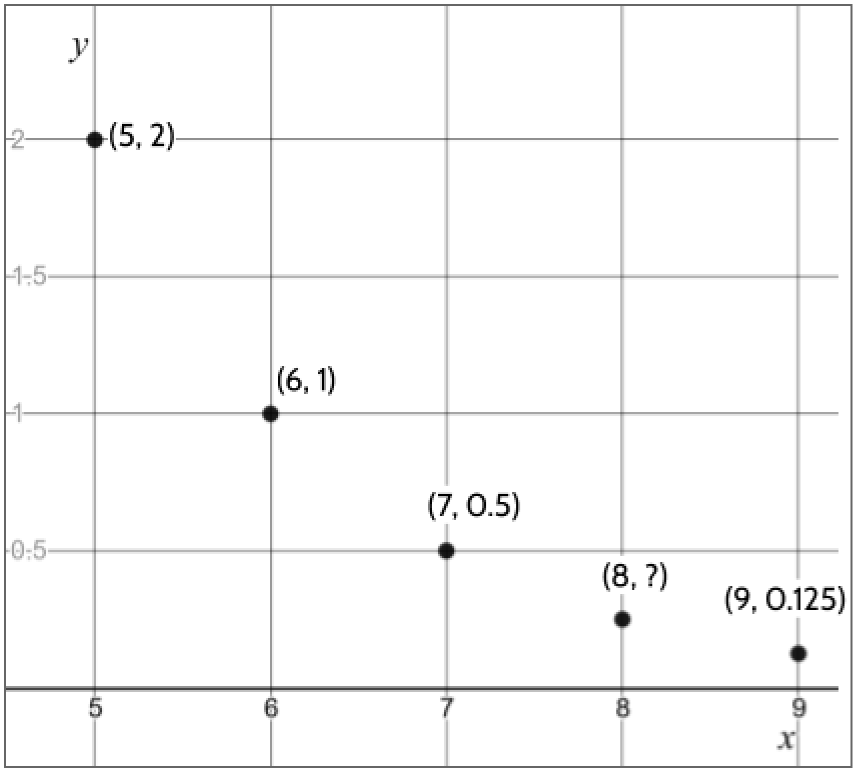
This graph shows more y values of $y = 1 \cdot 2^x$ as x increases to $x = 20$.



33) What might be surprising about this graph?

34) Can you explain why the dots are so close to the x-axis on the left side of the graph?

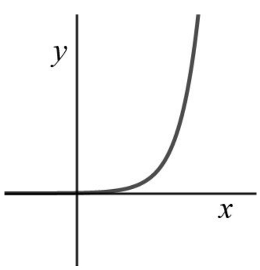
35) This graph shows more y values of $y = 64 \cdot (.5)^x$ as x increases.



What value of y is missing for an x value of 8?

36) As x increases, do you think the dots will go below $y = 0$? Why or why not?

37) Which of these graphs would the function $y = 1.2^x$ produce?



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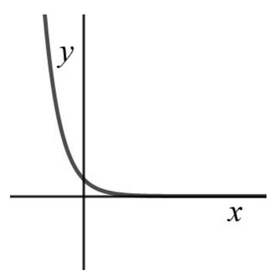


II.

Comparing Linear, Quadratic, and Exponential Functions

Are the following number sentences always true, sometimes true or never true?

Examine the expressions to make a decision. You should complete each function table in order to analyze the sentence.

38) x^2 is greater than $2x$.

x	-3	-2	-1	0	1	2	3	4	5
$2x$	−6	−4	−2	0	2	4	6	8	10
x^2	9	4	1	0	1	4	9	16	25

This number sentence is sometimes true. x^2 is greater than $2x$ when $x > 2$. For example, if $x = 3$, then $x^2 = 9$ and $2x = 6$. If $x = 2$, then $x^2 = 4$ and $2x = 4$ as well. If $x = 1$, $2x$ is greater than x^2, since $2(1) = 2$ and $1^2 = 1$.

39) $2x + 7$ is less than $4x + 3$

x	-3	-2	-1	0	1	2	3	4	5
$2x + 7$									
$4x + 3$									

40) $x^2 - 1$ is equal to $2x + 7$.

x	-3	-2	-1	0	1	2	3	4	5
$x^2 - 1$									
$2x + 7$									

41) $x^2 > 2^x$

x	-3	-2	-1	0	1	2	3	4	5
x^2									
2^x									

42) x^2 is less than x^3.

x	-3	-2	-1	0	1	2	3	4	5
x^2									
x^3									

43) If x is greater than 0, then $3^x > 2^x$.

x	0	1	2	3	4	5	6	7	8
3^x									
2^x									

Exponential Functions in the World

Choose Your Salary II

Earlier in the packet, we posed the following problem:

> I would like to offer you a job for thirty days. You will only need to work one hour each day. The interesting part is that I am going to let you decide how you want to be paid. If you like, I will pay you $1,000 per day. Your other option is to be paid 1 penny on the first day, 2 pennies on the second day, 4 pennies on the third day, and so on, each day earning twice as many pennies as the day before.

1) Which way of being paid did you choose originally? Do you still agree with your original choice? Please explain.

2) Using the $1,000-per-day method, what function would tell you the total amount of money you would make after n days?

n	$f(n)$
1	$1,000.00
2	$2,000.00
3	$3,000.00
4	$4,000.00
5	$5,000.00

3) Using the $1,000-per-day method, how much money would you make in one month (30 days)?

The total amount of money paid with the doubling-penny method is shown on the right. Amazingly, you would make a LOT more money with this method (if you could convince someone to pay you in this way).

4) Which method would you choose if I was only going to hire you for 10 days? Please explain.

5) Which method would you choose if the job lasted 25 days? Please explain.

6) After how many days does it become a better deal to choose the doubling-pennies method instead of the $1,000-per-day method?

Day	Payment
1	$0.01
2	$0.02
3	$0.04
4	$0.08
5	$0.16
6	$0.32
7	$0.64
8	$1.28
9	$2.56
10	$5.12
11	$10.24
12	$20.48
13	$40.96
14	$81.92
15	$163.84
16	$327.68
17	$655.36
18	$1,310.72
19	$2,621.44
20	$5,242.88
21	$10,485.76
22	$20,971.52
23	$41,943.04
24	$83,886.08
25	$167,772.16
26	$335,544.32
27	$671,088.64
28	$1,342,177.28
29	$2,684,354.56
30	$5,368,709.12
Total	**$10,737,418.23**

Going Viral

Dear Lacey,

Share this post[3] with five people today, or terrible bad luck will afflict you. The good luck of Flanders was sent to me and I am sharing it within twenty-four hours. This chain was started by an American Officer in Flanders and is going around the world four times and one who breaks it will have bad luck. Share this post and see what happens to you tomorrow. It will bring you good luck.

Ms. Sally Anderson received $5000, five hours after posting. Ms. Ambrose won the lottery, four hours after posting. Mr. Nevin broke the chain and lost everything he had.

Please do not break the chain! Repost this message to five people now!

Henry

7) Have you ever received a post like this on social media or through email? Did you forward the message to other people?

8) If Lacey shares the post with 5 friends on Day 1 and those friends each share the post with 5 of their friends on Day 2, how many people will receive the post on the 2nd day?

[3] This post is based on a chain letter that was started in 1939 and was sent through the mail. Chain letters attempts to convince people to send the message to other people. They sometimes promise good luck or money. They can also be used to take advantage of people who share personal information. If a chain letter asks for money, it is probably against the law. People sending chain letters may be trying to commit fraud.

9) Complete the table for the first ten days.

Day #	Calculation	Number of people receiving post
1	$1 \cdot 5$	5
2	$1 \cdot 5 \cdot 5$	25
3		
4		
4		
5		
6		
7		
8		
9		
10		

10) How many days until the number of posts received that day is greater than the population of the United States (about 330 million people in 2019)?

11) Which expression shows the number of posts received on the 10th day?

A. $5(10)$

B. 10^5

C. 5^{10}

D. $\frac{10}{5}$

12) Which expression shows the number of posts received on the *n*th day?

A. $5n$

B. n^5

C. 5^n

D. $\frac{n}{5}$

13) If each person shared the post six times instead of five, how would your answer to the last question change?

The number of people receiving the social media post grows by a factor of 5 each day: 5, 25, 125, 625, etc. This is an example of *exponential growth*[4]. The number of people receiving social media posts can grow *exponentially*, since each person who receives an interesting or exciting or controversial post is likely to share it with a few of their friends, who are also likely to share it with their friends. If this increase continues, it is called "going viral."

14) What are some other examples of social media posts spreading quickly?

15) The word *viral* is from the word *virus*, which is a microscopic organism that can make you sick. What do you think exponential growth has to do with getting sick?

[4] You can find more practice with exponential growth in **The Power of Exponents** packet.

Growing Bacteria

E. coli (Escherichia coli) is a type of bacteria that lives in the digestive tracts of humans and animals. A population of E. coli can double every 20 minutes. These photographs show a population of E. coli bacteria in a petri dish. The photographs were taken every 20 minutes. 01:20:00 means that 1 hour and 20 minutes have passed since the bacteria started growing.

In the first photo, there is 1 bacteria. After 20 minutes, the bacteria split into 2 bacteria. After another 20 minutes, the 2 bacteria split again. And so on.

16) How many bacteria are there after 1 hour?

17) How many bacteria will there be after 1 hour and 40 minutes?

18) Do you think the following statement is True or False? Explain your answer.
Starting with just one E. coli bacteria, the population can grow to more than a million bacteria in one day.

19) E. coli can double in population every 20 minutes. Use the table to find out how many 20-minute increments are in h hours.

Hours	20-minute increments
1	3
2	
3	
4	
h	

If the bacteria have enough food and space, the population will multiply by 2 every 20 minutes. You can use exponents to predict the size of the population after *n* number of 20-minute increments pass.

20) Complete the table below to determine the amount of E. coli bacteria given the time that has passed.

Time	Number of 20-min increments (*n*)	Number of bacteria	Number of bacteria (exponential form)
0:00:00	0	1	2^0
0:20:00	1	2 = $1 \cdot 2$	2^1
0:40:00	2	4 = $1 \cdot 2 \cdot 2$	2^2
1:00:00	3	8 = $1 \cdot 2 \cdot 2 \cdot 2$	
1:20:00	4	16 =	2^4
1:40:00	5	=	
2:00:00	6	=	
2:20:00		=	
	8	=	
	9	=	
	n		2^n

21) How many bacteria will there be after 4 hours?

22) How much time would it take to grow a million bacteria?

23) How many bacteria will there be after 24 hours?
(Your answer can be in exponent form)

If you want to see a time-lapse video of bacteria growing, go to http://bit.ly/bacteriagrowth.

Rumors and Measles

A rumor is a story or a piece of information that is spread from person to person. Rumors may or may not be true.

24) When was the last time you heard a rumor?

25) How do rumors spread through a population of people?

Suppose on Day 1, a single person tells someone else a rumor, and on the next day, each person who knows the rumor tells it to exactly one other person. And then the following day everyone who knows the rumor tells one other person. And so on.

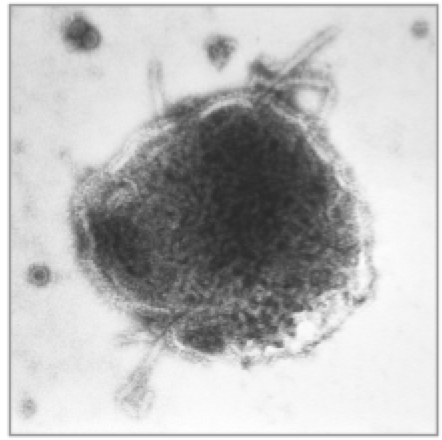
26) Continue the table.

Day	How many people are told the rumor during the day?	How many people know the rumor by the end of the day?
1	1	2
2	2	4
3		
4		
5		
6		

27) How many days would pass before 50 people have heard the rumor?

28) How days would pass before 100 people have heard the rumor?

The number of people who have heard the rumor doubles every day. This happens because each day, every person who knows the rumor tells it to a new person. Twice as many people are told every day.



measles virus

In some ways, this is similar to how a contagious disease like the flu, chicken pox, or measles can pass through a population of people. People who are infected with the disease can pass it to others.

In our rumor example, everyone who was told the rumor became "contagious." In other words, there was a 100 percent transmission rate. 100 percent of those who know the rumor spread it to someone else.

The number of people "infected" by the rumor increases by a *growth rate* of 100%. Adding 100% more people each day is the same as multiplying by a *growth factor* of 2. Each day, the growth rate of 100% is added to 100% of the original "infected" population. If all the numbers are kept in percent, the growth rate plus the original amount equals the growth factor.

29) Complete the table to convert growth rate into a growth factor.[5]

Original Amount		Growth Rate		Growth Factor	
Percentage	Decimal	Percentage	Decimal	Percentage	Decimal
100%	1	100%	1	200%	2
100%	1	90%	0.9	190%	1.9
100%	1	80%	0.8		
100%	1	45%		145%	
100%	1	20%			
100%	1	5%	0.05		
100%	1	1%			

[5] You can find more growth rate and growth factor practice in **The Power of Exponents.**

30) Here is one way of looking at the rumor transmission at a 100% growth rate. Complete the table.

Day	Multiplying by Growth Factor	Exponent	Total Number of "Infections"
1	$1 \cdot 2$	2^1	2
2	$1 \cdot 2 \cdot 2$	2^2	4
3	$1 \cdot 2 \cdot 2 \cdot 2$	2^3	
4	$1 \cdot 2 \cdot 2 \cdot 2 \cdot 2$		
5			
n		2^n	

31) Expressed as an exponent, how many "infections" would you expect by day 10?

A. 10^2

B. 10^n

C. 2^{10}

D. 2

32) Expressed as a numeral, how many "infections" would you expect by day 10?

A. 100

B. 1024

C. 20

D. 200

The measles virus is transmitted by direct contact with an infected person or spread when an infected person breathes, coughs, or sneezes. The measles virus can remain infectious in the air for up to two hours after an infected person leaves an area. Some diseases are more infectious than others, but usually the transmission or growth rate isn't 100%. Measles is a very contagious disease. If a population is unvaccinated and hasn't acquired immunity, about 90% of people exposed to the virus can become infected. This means that 9 out of 10 unvaccinated people who are infected develop the disease.

Suppose on Day 0, one person has measles. On Day 1, the person infected with measles is in contact with another person who is unvaccinated. Since measles has a transmission rate of 90%, there is a 90% chance that the second person will get measles. On subsequent days, people who are infected can pass the disease to people who aren't vaccinated.

33) Continue the table.

Day	Multiplying by Growth Rate	Exponent	Total Number of Infections (Round to whole number)
1	$1 \cdot 1.9$	1.9^1	1.9 → 2
2	$1 \cdot 1.9 \cdot 1.9$	1.9^2	3.6... → 4
3	$1 \cdot 1.9 \cdot 1.9 \cdot 1.9$		6.8... →
4			
5			
n			

34) How many people could be infected by Day 5?

35) How many people could be infected by Day 8?

36) Which expression represents the total number of people infected by the measles virus by Day 10?

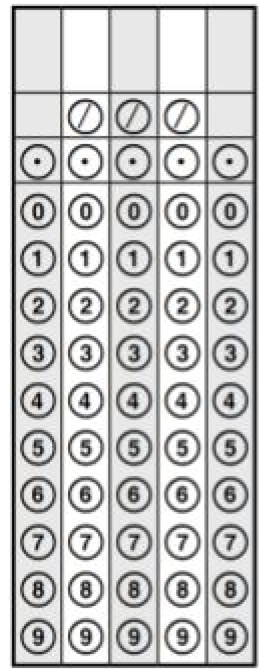
A. 1.9^{10}

B. $10^{1.9}$

C. $10 \cdot 1.9$

D. $(10)(1.9)$

37) What is the total number of people with measles by Day 10? Round to the nearest whole number and write your answer in the grid on the right.



38) The number of people infected by the measles virus in an unvaccinated population can be modeled by the function $M(t) = (1.9)^t$, where $M(t)$ is the number of infected people and t is the time in days.

Part 1:

What is the percent change in the infected population, and does this represent exponential growth or decay?

A. 9%; exponential decay

B. 9%; exponential growth

C. 90%; exponential decay

D. 90%; exponential growth

39) If one person gets infected with a virus that has a transmission rate of 50%, how long might it take for 1,000 people to get infected if no one has been vaccinated?

A. 18 days

B. 667 days

C. 1,000 days

D. 2,000 days

What is a vaccine?

A *vaccine* is a medical treatment which makes the body stronger against a particular germ (bacteria or virus). For example, the flu vaccine makes it less likely that a person will get influenza (the flu).

According to the Centers for Disease Control and Prevention (CDC), about 700 people in the United States were sickened by measles in the first five months of 2019. More than 500 of the people infected were not vaccinated. Vaccines are very effective in preventing people from becoming infected, even if they are in contact with someone who has measles. If someone has been vaccinated against the measles virus, there is a 97% chance that they will not get measles if they are exposed to the virus.

The body fights infections using the immune system, which is a network of cells, tissues, and organs that work together to protect the body. When the immune system sees an invading virus or bacteria, it responds by producing *antibodies* which find the invading cells and tell the body how to defend against them.

Once produced, these antibodies stay in a person's body, so that if their immune system encounters the same virus or bacteria again, the antibodies are already there to do their job. So if someone gets sick with a certain disease, like chickenpox, that person usually won't get sick from it again. In other words, they are now *immune* to chickenpox. The immune system is much stronger when fighting a disease it has fought against before.

A vaccine is a way of becoming immune to a disease without actually getting the disease in the first place. The vaccine is made of cells that look like the original virus and bacteria, which encourages the body to develop antibodies to fight future infections.

Herd immunity is an important part of how vaccines work. A herd is a group of animals. Herd immunity happens when most of the animals in a group are immune to an infection. If animals are immune, they cannot get the disease. Since they do not get the disease, they cannot give it to other animals, so even an animal who is not immune is safer. If none of the other animals in a herd get the infection, they cannot give the infection to the one who is non-immune. Herd immunity happens when enough of the animals are immune so that the whole herd is protected from a disease.

This is important in people too. Imagine a population where 95% of the people are immune to measles and 5% are non-immune. Some of the 5% may have gotten the vaccine but did not react to it. Some of them may have been too sick to get the vaccine. Since 95% of people in a place are immune to a disease, the 5% who are non-immune are safer. There will just not be as much of the disease to be exposed to.

However, if fewer people in the population are vaccinated, it means that more people will get infected and will pass the infection to other people. With a highly contagious disease like measles, it is dangerous for people to be non-vaccinated. If they are not protected by herd immunity, individuals can get the infection more easily. So it is important that people who are healthy get their vaccinations. If most people in a place get vaccinated, everyone has more protection. Vaccination protects healthy people, but it also protects children and other people who are old, weak, or sick.

In response to the record number of measles cases in 2019, New York City imposed a mandatory vaccination order, and some New York counties required that anyone with measles avoid public spaces.

(Adapted from Wikipedia.org)

Allergy Medication

Every morning, Ramon takes 20 milligrams of a medication to help with his allergies. As time goes by, his body *metabolizes* the medicine so that there is less and less in his body later in the day. Our bodies metabolize substances such as food, medicine, and alcohol. *Metabolize* means to break down, absorb and use a substance. When a substance is metabolized, it disappears over time.

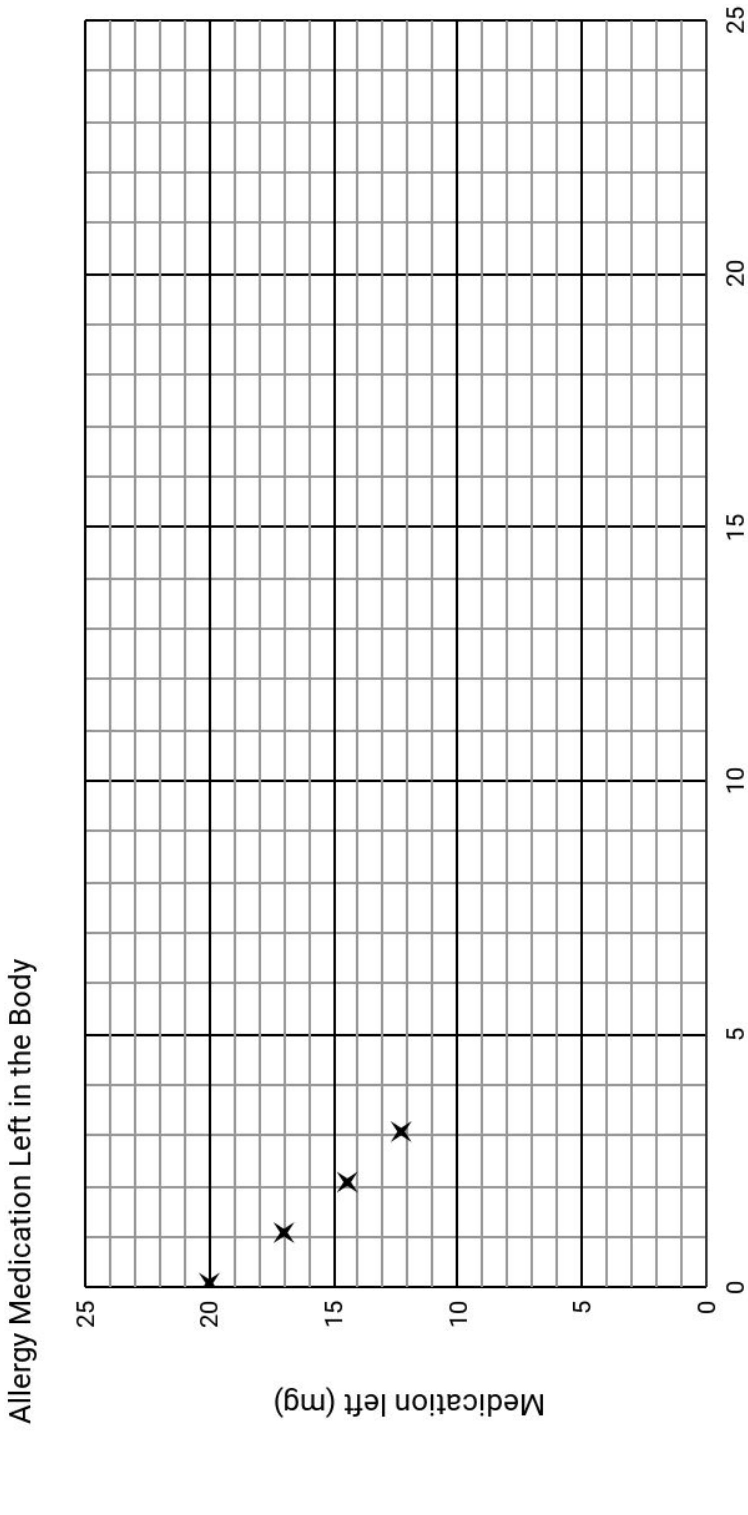
The amount of medication left in Ramon's body after metabolization can be modeled by the function $f(t) = 20 \cdot (0.85)^t$, where *t* stands for *time* in hours.

40) Complete the table.

Hour t	Formula $20 \cdot (0.85)^t$	Calculation	Medication left (mg) (Round to tenths place) $f(t)$
0	$20 \cdot (0.85)^0$	20 · 1	20
1	$20 \cdot (0.85)^1$	20 · 1 · 0.85	17
2	$20 \cdot (0.85)^2$	20 · 1 · 0.85 · 0.85	14.45 → 14.5
3	$20 \cdot (0.85)^3$		12.28... →
4			
5			

41) How many milligrams of allergy medication is in Ramon's body after 8 hours?

42) Complete the table and graph.



Complete the table using the function $f(t) = 20 \cdot (.85)^t$, then plot the points on the graph. Some have been done for you.

t	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24
$f(t)$	20	17	14.5	12.3	10.4	8.9			5.4																

43) What do you notice when looking at the Allergy Medication Left in the Body graph?

44) Does the function for allergy medication produce exponential growth or decay? Why?

45) If Ramon started taking 5 milligrams less allergy medication each day, which function would model the metabolization of the medication?

A. $f(t) = 15 \cdot (0.80)^t$

B. $f(t) = 20 \cdot (0.80)^t$

C. $f(t) = 15 \cdot (0.85)^t$

D. $f(t) = 20 \cdot (0.90)^t$

Test Practice Questions

Answer the following questions. You can check your answers in the answer key at the end of the packet.

1) These are the first four terms in a number sequence: 1, 3, 6, 10.
If you continue the sequence, what is the 7th term?

A. 26

B. 27

C. 28

D. 29

2) The input/output table on the right represents a quadratic function.

Part I: What is the missing value for y?

A. 24

B. 26

C. 27

D. 34

Part II: Which equation matches the table?

A. $2x^2 - 1 = y$

B. $2x^2 + 1 = y$

C. $x^2 - 1 = y$

D. $x^2 + 1 = y$

x	y
0	1
1	2
2	5
3	10
4	17
5	

3) These are the first four terms in a number sequence: 3, 6, 12, 24. If you continue the sequence, what is the 5th term?

A. 30

B. 36

C. 42

D. 48

4) Which ordered pair is <u>not</u> a solution to $y = x^2 + 2x + 1$?

A. $(-1, 0)$

B. $(0, 1)$

C. $(3, 12)$

D. $(5, 36)$

5) The table shows values of a function $f(x)$.

x	$f(x)$
1	4
2	7
3	
4	13
5	16

Part I: What is the value for $f(3)$?

A. 8

B. 9

C. 10

D. 11

Part II: What kind of function is $f(x)$?

A. rational

B. exponential

C. linear

D. quadratic

6) The table shows values of a function $g(x)$. What kind of function is $g(x)$?

x	$g(x)$
0	0
1	2
2	8
3	18
4	32

A. rational

B. exponential

C. linear

D. quadratic

7) The table shows values of a function $h(x)$. Which type of function best models the given data?

x	$h(x)$
0	1
1	3
2	9
3	27
4	81

A. linear function with negative rate of change

B. linear function with positive rate of change

C. exponential growth function

D. exponential decay function

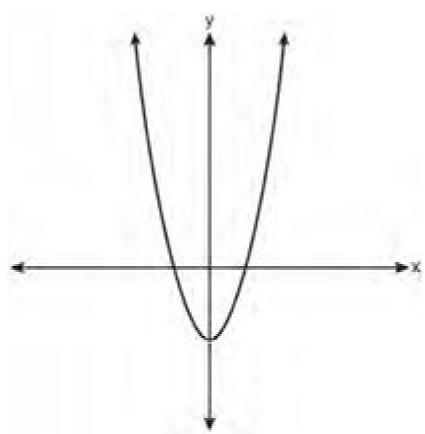
8) Which type of function is represented by the graph shown to the right?

A. rational

B. exponential

C. linear

D. quadratic



9) If $f(x) = 3x^2 - 2x + 1$, what is $f(2)$?

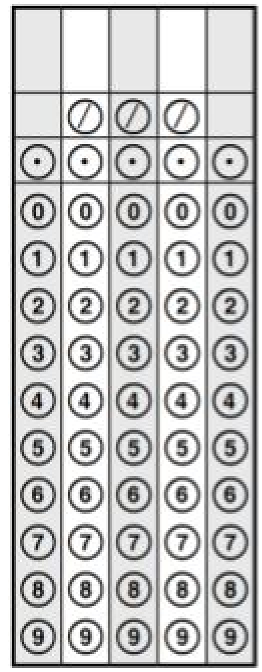
A. 1

B. 2

C. 6

D. 9

10) Consider the equation $x^2 = 25$. For $x \geq 0$, what is the value of x? Enter your answer in the grid to the right.



11) The table shows values of a function $f(x)$.

Determine the average rate of change of the function from $f(3)$ to $f(6)$.

A. -1

B. 5

C. 7

D. 14

x	$f(x)$
2	-2
3	-1
4	2
5	7
6	14
7	23

12) An economist recorded the stock price of Company A after the initial stock sale.

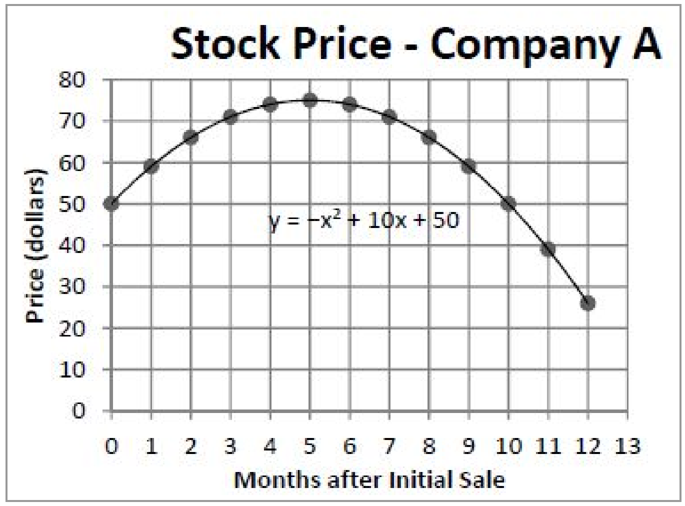
What was the value of the stock price of Company A 10 months after the initial stock sale?

A. 25

B. 40

C. 50

D. 75



13) Which expression is the product of $(x+4)$ and $(x-2)$?

A. x^2+6x-8

B. x^2-6x-8

C. x^2+2x-8

D. x^2+2x+8

14) Which of the following is the factored form of x^2+4x+3?

A. $(x+1)(x+3)$

B. $(x+1)(x-3)$

C. $(x-1)(x+3)$

D. $(x-1)(x-3)$

15) Which equation can be used to find the solutions of $y=x^2-3x-10$?

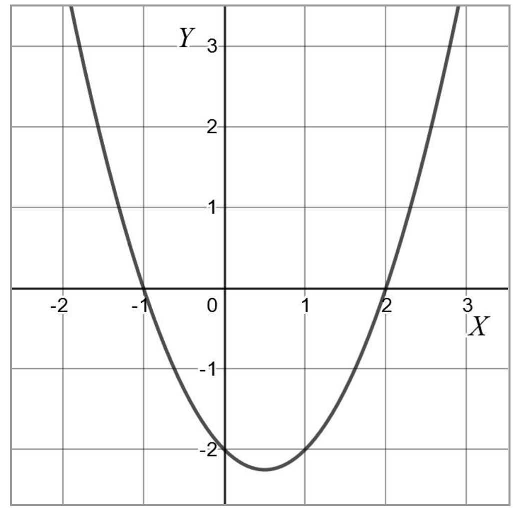
A. $(x-2)(x+5)=0$

B. $(x+2)(x-5)=0$

C. $(x+3)(x-10)=0$

D. $(x+3)(x+10)=0$

16) What are the zeros in the graph of the function $f(x)=(x+1)(x-2)$?



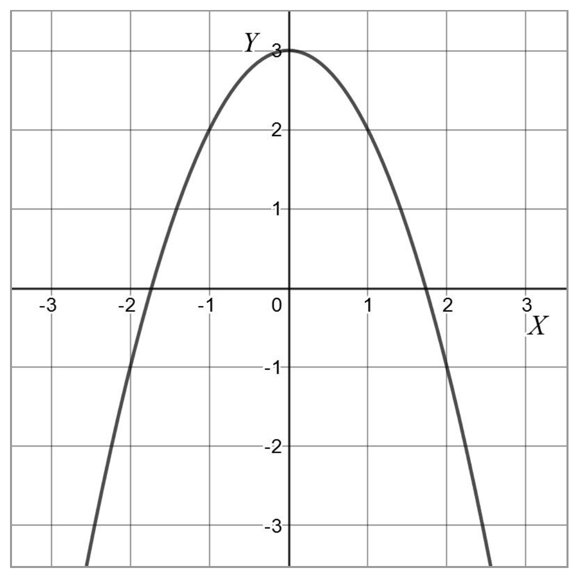
A. $x = 1$ or $x = -2$

B. $x = -1$ or $x = -2$

C. $x = 1$ or $x = 2$

D. $x = -1$ or $x = 2$

17) Which equation best matches the graph?



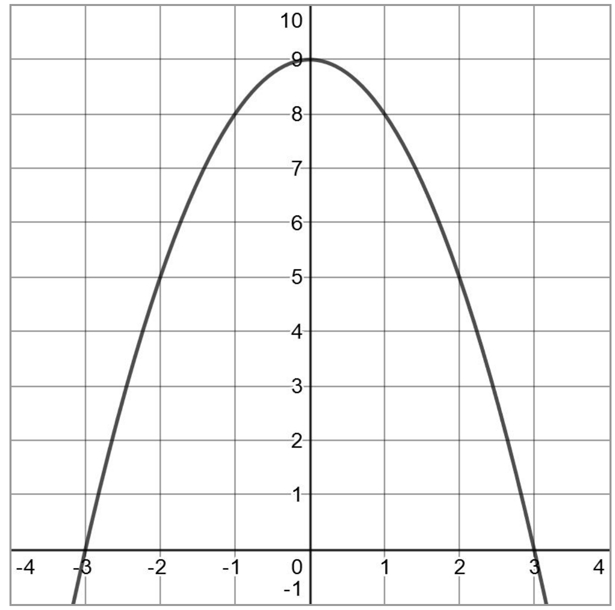
A. $y = -x^2 + 3$

B. $y = x^2 + 3$

C. $y = -x^2 + 2$

D. $y = x^2 + 2$

18) Consider the graph of the polynomial function $y = -x^2 + 9$ as shown.



What is the positive zero of this polynomial function?

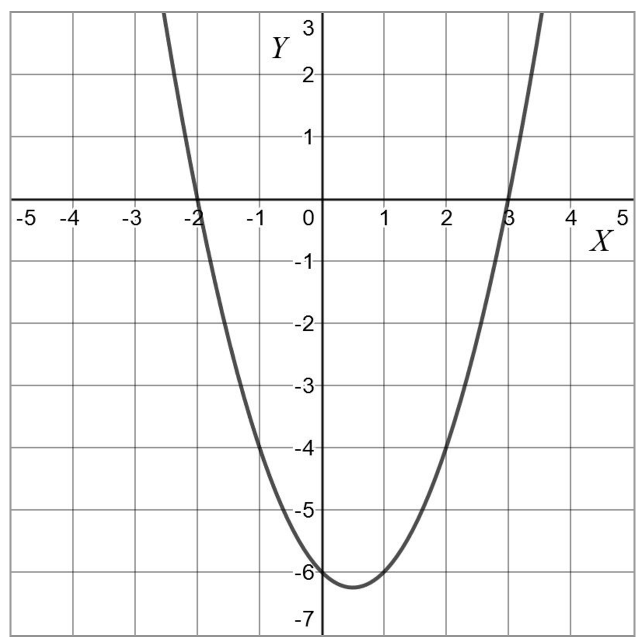
A. -3

B. 0

C. 3

D. 9

19) What are the roots of the quadratic equation associated with the graph?



A. -6 and 3

B. -6 and 0

C. -3 and 2

D. -2 and 3

20) The function $P(t) = 50,000 \cdot 1.01^t$ models the change in a population over time, starting with an initial population of 50,000. *P*(*t*) is the size of the population after *t* years.

Which total is closest to the predicted population after 10 years?

A. 45,000

B. 50,500

C. 55,000

D. 505,000

21) Which function equation matches the table on the right?

A. $y = x^2 + 8$

B. $y = x^2 - x$

C. $y = x^2 + 4x$

D. $y = x^2 + 8x + 16$

x	y
0	0
1	5
2	12
3	21
4	32
5	45

22) Which table displays an exponential function?

A.

x	1	2	3	4	5
y	2	4	6	8	10

B.

x	1	2	3	4	5
y	4	16	32	48	64

C.

x	1	2	3	4	5
y	3	6	9	12	15

D.

x	1	2	3	4	5
y	2	4	8	16	32

23) Consider the equations below.

$y = x^2 - 1$

$y = 2x + 7$

Part I: Which ordered pair is a solution to the system of equations?

A. (4, 15)

B. (1, 0)

C. (15, 4)

D. (0, 7)

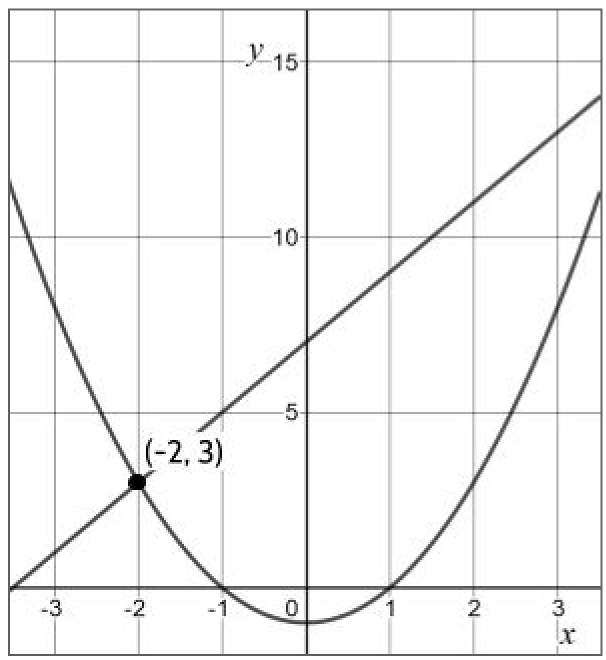
Part II: Why is the coordinate pair (-2, 3) important?

A. It isn't considered a solution because x is negative.

B. It is the only solution to the system of equations.

C. It shows the 2nd solution to the system of equations.

D. It is the only place where the line and the parabola intersect.



24) What are the factors of the expression $x^2 + x - 20$?

A. $(x+5)$ and $(x+4)$

B. $(x+5)$ and $(x-4)$

C. $(x-5)$ and $(x+4)$

D. $(x-5)$ and $(x-4)$

25) The height of a water-powered rocket at time t seconds is given by $h(t) = -6t^2 + 96t$.

$t = time$
$h = height\ in\ feet$

How high will the rocket be after 2 seconds?

A. 6 feet high

B. 96 feet high

C. 132 feet high

D. 168 feet high

26) Which situation could be modeled by using a linear function?

A. a bank account balance that grows at a rate of 5% per year, compounded annually

B. the cost of cell phone service that charges a base amount plus 20 cents per minute

C. a population of bacteria that doubles every 4.5 hours

D. the concentration of medicine in a person's body that decays by a factor of one-third every hour

27) Which scenario represents exponential growth?

    A. A water tank is filled at a rate of 2 gallons/minute.

    B. A species of fly doubles its population every month during the summer.

    C. A vine grows 6 inches every week.

    D. A car increases its distance from a garage as it travels at a constant speed of 25 miles per hour.

28) Lacey is saving up to buy a new phone. Every week she puts $20 into a jar. Which type of function best models the total amount of money in the jar after a given number of months?

    A. linear

    B. quadratic

    C. exponential

    D. square root

29) The table below shows the end of the year balance in an investment account where interest is compounded annually. No money is deposited or withdrawn after the initial amount is deposited.

Year	Balance in dollars
0	$500.00
1	$525.00
2	$551.25
3	$578.81
4	$607.75
5	$638.14

Which type of function best models the given data?

A. linear function with negative rate of change

B. linear function with positive rate of change

C. exponential growth function

D. exponential decay function

30) The length of a rectangular window is 5 feet more than its width, w. The area of the window is 36 square feet. Which equation could be used to find the dimensions of the window?

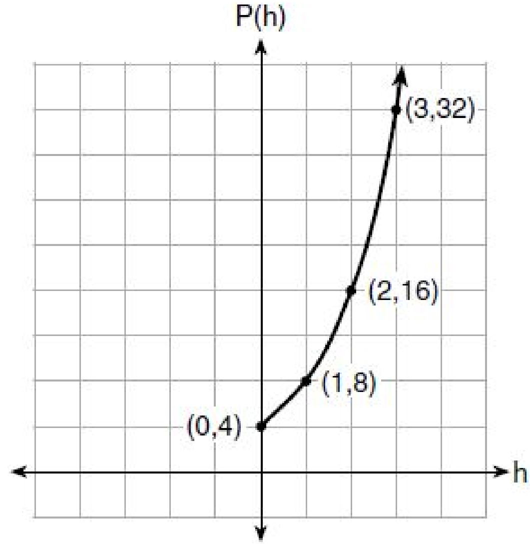
A. $w + 5w = 36$

B. $5w^2 + w = 36$

C. $w^2 + 5 = 36$

D. $w^2 + 5w = 36$

31) Liz collected population data, $P(h)$, from a colony of E. coli bacteria over time in hours, h, as shown in the graph.



Part I: Which equation matches the data in the graph?

A. $P(h) = 4h + 4$

B. $P(h) = 4h^2$

C. $P(h) = 4 \cdot 2^h$

D. $P(h) = 2 \cdot 4^h$

Part II: Over how many hours did Liz collect population data?

A. 3

B. 4

C. 16

D. 32

32) What is the percent change in the population modeled by $P(t) = 5,000 \cdot 1.01^t$, and is this exponential growth or decay?

A. 1%; exponential decay

B. 1%, exponential growth

C. 101%, exponential decay

D. 101%, exponential growth

33) The highest possible grade for a book report is 100. The teacher deducts 10 points for each day the report is late. Which kind of function describes this situation?

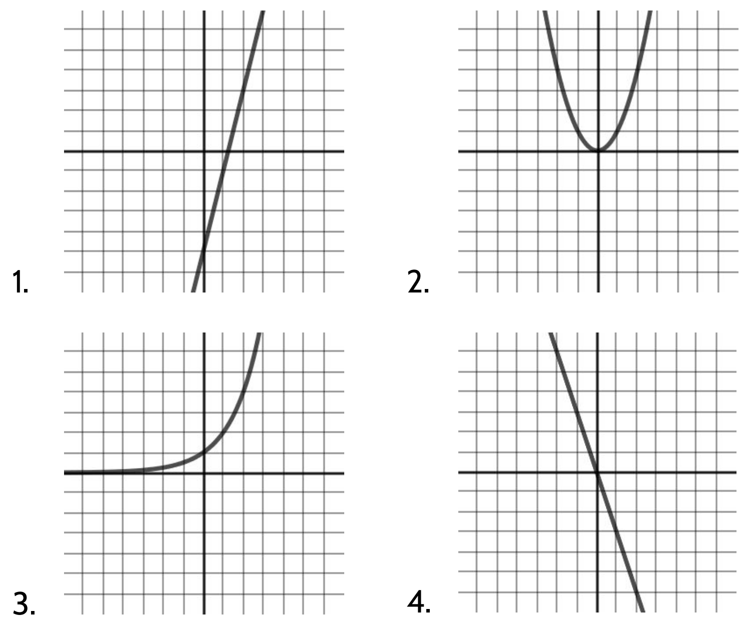
A. linear

B. exponential growth

C. quadratic

D. exponential decay

34) Look at the graphs and then answer the questions below.



Part I: Which statement describing these graphs is false?

A. Graphs 1 and 4 represent linear functions.

B. Graphs 2 and 3 represent nonlinear functions.

C. Graph 2 is a quadratic function.

D. Graph 3 shows a constant rate of change.

Part II: Which graph shows exponential growth?

A. 1

B. 2

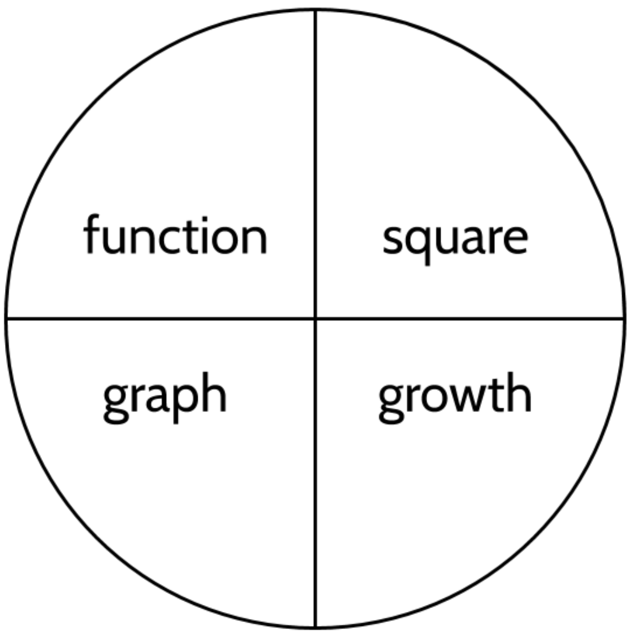
C. 3

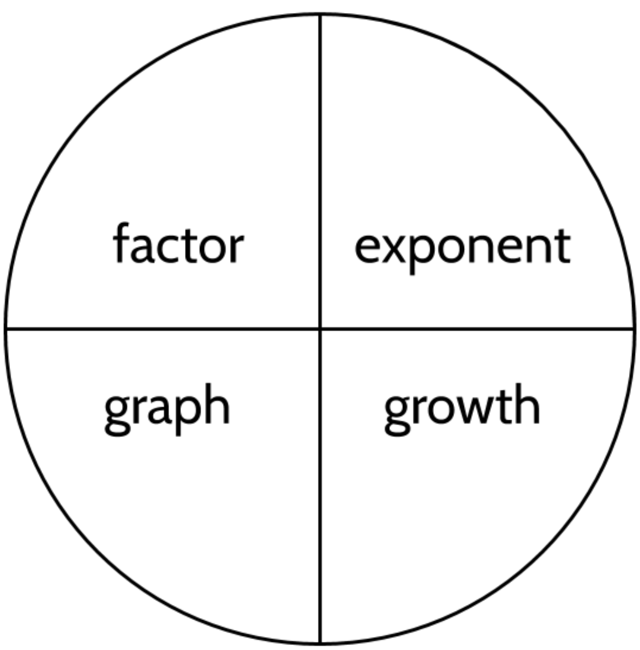
D. 4

The Language of Expressions, Equations, and Inequalities

Concept Circle

1) Explain these words and the connections you see between them.





Nonlinear Functions in the World

2) Look around you. Where do you see quadratic and exponential functions? Describe an example from the world using vocabulary words from the end of the packet.

Answer Keys

Exploring Quadratic Growth II

1) The acceleration (change in speed) is +2 for the whole table.

time (seconds)	distance (feet)	speed (change in distance)	acceleration (change in speed)
0	0		
1	1	+ 1	
2	4	+ 3	+ 2
3	9	+ 5	+ 2
4	16	+ 7	+ 2
5	25	+ 9	+ 2
6	36	+ 11	+ 2
7	49	+ 13	+ 2
8	64	+ 15	+ 2

2) We're interested in what you notice.

3) Write at least 2-3 questions. We share ours on the following page.

4) Answers will vary.

5) Answers will vary.

6) 5 meters down. That's why the number is negative.

7) 15 meters down.

8) The ball is speeding up as it falls.

9) What do you predict? Explain how you came up with those numbers.

10)

time (seconds)	distance (meters)	rate of change
t	$d(t)$	
0	0	
		-5
1	-5	
		-15
2	-20	
		-25
3	-45	
		-35
4	-80	
		-45
5	-125	
		-55
6	-180	
		-65
7	-245	
		-75
8	-320	
		-85
9	-405	
		-95
10	-500	

11) 500 meters

12) Answers will vary.

13) -10

14)

time (seconds)	distance (meters)
t	$d(t)$
0	0
1	-5
2	-20
3	-45
4	-80
5	-125
6	-180
7	-245
8	-320
9	-405
10	-500

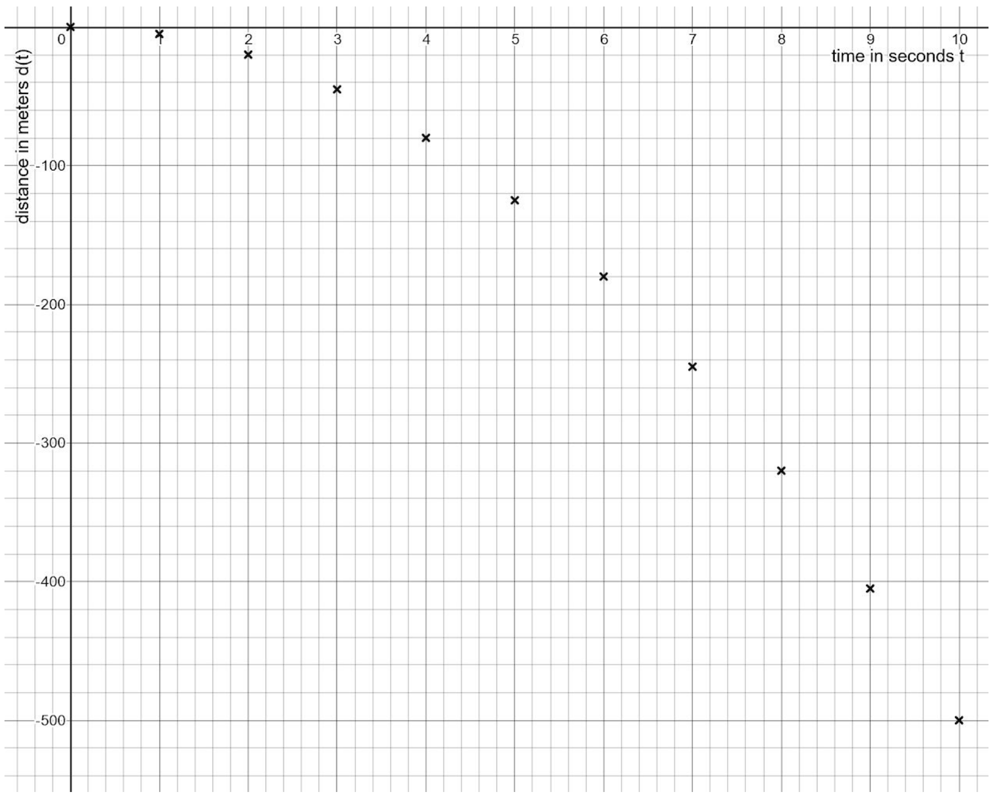
average speed (rate of change)

-5, -15, -25, -35, -45, -55, -65, -75, -85, -95

acceleration (change in the rate of change)

-10, -10, -10, -10, -10, -10, -10, -10, -10

15) -30. The ball has traveled -180 feet in 6 seconds. It's not moving at a constant speed, but the average speed over this time period is -180 ÷ 6 = -30.

16) 
0
1
2
3
4
5
6
7
8
9
10
time in seconds t
distance in meters d(t)
-100
-200
-300
-400
-500

17)

time (seconds)	distance (feet)
t	$d(t)$
0	0
1	-16
2	-64
3	-144
4	-256
5	-400
6	-576
7	-784
8	-1024
9	-1296
10	-1600

-16, -48, -80, -112, -144, -176, -208, -240, -272, -304

-3, -3, -3, -3, -3, -3, -3, -3, -3

B.

18)

time (seconds)	distance (meters)
t	$d(t)$
0	0
1	-4.9
2	-19.6
3	-44.1
4	-78.4
5	-122.5
6	-176.4
7	-240.1
8	-313.6
9	-396.9
10	-490.0

-4.9, -14.7, -24.5, -34.3, -44.1, -53.9, -63.7, -73.5, -83.3, -93.1

-9., -9., -9., -9., -9., -9., -9., -9., -9.

B.

19)

$f(x) = \frac{1}{2}(-32)x^2$

$f(15) = \frac{1}{2}(-32)\cdot(15)^2$

$f(15) = (-16)\cdot 225$

$f(15) = -3600\ feet$

$f(x) = \frac{1}{2}(-9.8)x^2$

$f(15) = \frac{1}{2}(-9.8)\cdot(15)^2$

$f(15) = (-4.9)\cdot 225$

$f(15) = -1102.5\ meters$

20) 11 seconds

21)

meter	Convert meters to feet	feet
1	1 × 3.28	3.28
2	2 × 3.28	6.56
6	6 × 3.28	19.68
10	10 × 3.28	32.8
5	5 × 3.28	16.4
.5	.5 × 3.28	1.64
7.5	7.5 × 3.28	24.6
9	9 × 3.28	29.52
9.8	9.8 × 3.28	32.144

This row shows that 9.8 meters = 32.144 feet.

9.8	9.8 × 3.28	32.144

22) 9.8 meters is about 2 inches longer than 32 feet.

23) $f(t) = \frac{1}{2}(-32)t^2$

$f(25) = \frac{1}{2}(-32)\cdot(25)^2$

$f(25) = (-16)\cdot 625$

$f(25) = -10000\,feet$

The ball fell 10,000 feet in 25 seconds, so the plane must be 10,000 feet in the air.

24)

time (seconds)	function calculation	height (feet)
t	$h(t) = 64t - 16t^2$	h(t)
0	$64 \times 0 - 16 \times 0^2$ $0 - 0$	0
.5	$64 \times .5 - 16 \times .5^2$ $32 - 16 \times .25$ $32 - 4$	28
1	$64 \times 1 - 16 \times 1^2$ $64 - 16$	48
1.5	$64 \times 1.5 - 16 \times 1.5^2$ $96 - 16 \times 2.25$ $96 - 36$	60
2	$64 \times 2 - 16 \times 2^2$ $128 - 16 \times 4$ $128 - 64$	64
2.5	$64 \times 2.5 - 16 \times 2.5^2$ $160 - 16 \times 6.25$ $160 - 100$	60
3	$64 \times 3 - 16 \times 3^2$ $192 - 16 \times 9$ $192 - 144$	48
3.5	$64 \times 3.5 - 16 \times 3.5^2$ $224 - 16 \times 12.25$ $224 - 196$	28
4	$64 \times 4 - 16 \times 4^2$ $256 - 16 \times 16$ $256 - 256$	0
4.5	$64 \times 4.5 - 16 \times 4.5^2$ $288 - 16 \times 20.25$ $288 - 324$	-36

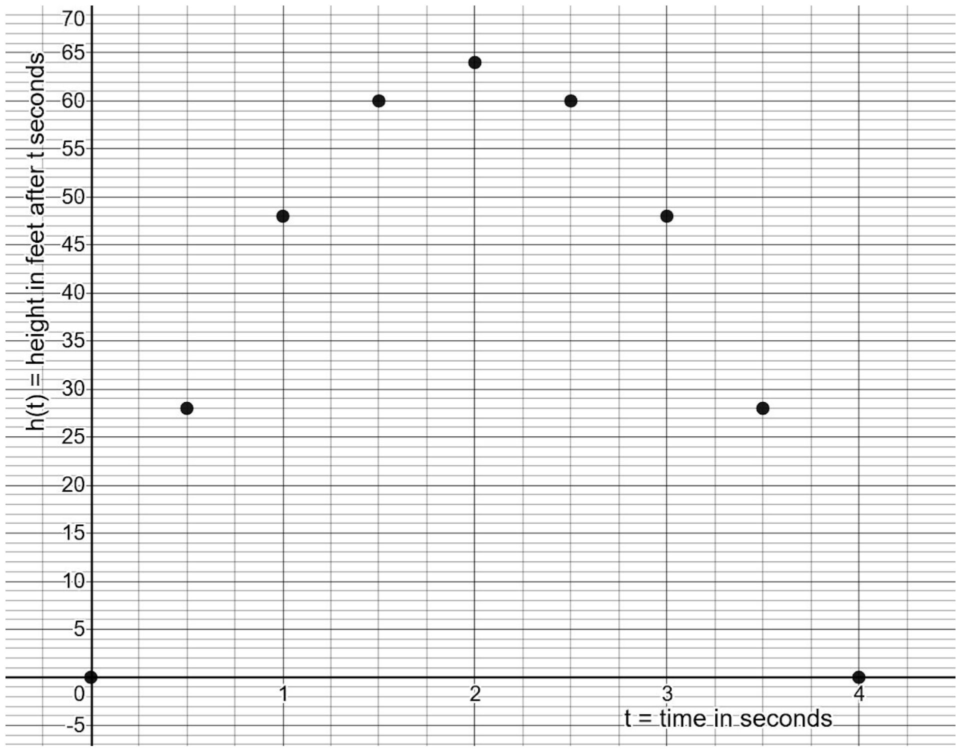
25) 48 feet

26) Because the ball is going up and then coming back down. The ball is at the same height at two different times.

27) 64 feet

28) 4 seconds. It hits the ground after 4 seconds.

29) -36 feet would mean that the water balloon has fallen below ground 36 feet. The balloon hits the ground after 4 seconds, so it has already stopped falling.

30) 

31) We're interested in what you notice.

32) (.5, 28) & (3.5, 28)

(1, 48) & (3, 48)

(1.5, 60) & (2.5, 60)

33) Because it happened after the water balloon hit the ground.

34) 5.5 feet

35) 16.5 feet

36) Answers will vary.

37) The car traveled 137.5 feet in 5 seconds, so the average rate of change is 27.5 feet per second ($13.5 \div 5 = 27.5$).

38) 550 feet

39) B

40) 352

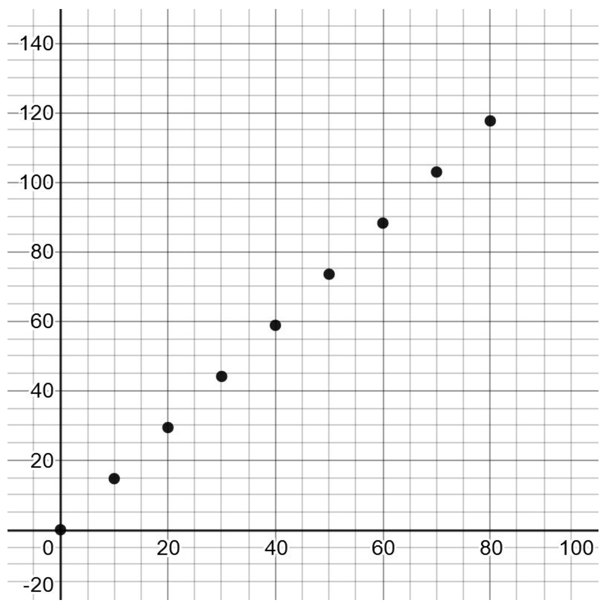
41) B

42) 73.5 feet per second

43)

mph	fps
10	14.7
20	29.4
30	44.1
40	58.8
50	73.5
60	88.2
70	102.9
80	117.6

44)



45) 1.47 feet per second

46) Answers will vary.

47) Answers will vary.

48)

Speed (mph)	Speed (fps)	Calculation	Reaction Distance (feet)
10	14.7	14.7 × 1.5	22.05
20	29.4	29.4 × 1.5	44.1
30	44.1	44.1 × 1.5	66.15
40	58.8	58.8 × 1.5	88.2
50	73.5	73.5 × 1.5	110.25

49) A

50) These are the points that are already plotted in the graph. It is possible to calculate of points as well.

Speed (mph)	Calculation	Braking Distance (feet)
10	$10^2 \div 20$	5
0	$0^2 \div 20$	0
20	$20^2 \div 20$	20
30	$30^2 \div 20$	45
40	$40^2 \div 20$	80

51) Nonlinear. Each input is squared (taken to the 2nd power), which is the definition of a quadratic growth pattern. You can also see that the graph is not linear and the outputs don't grow at a constant rate.

52) No. If you double your speed from 20 to 40 miles per hour, your braking distance increases from 20 feet to 80 feet (quadruples or is multiplied by 4).

53)

Speed (mph)	Reaction Distance (feet)	Braking Distance (feet)	Total Stopping Distance (feet)
10	22.05	5	27.05
20	44.1	20	64.1
30	66.15	45	111.15
40	88.2	80	168.2
50	110.25	125	235.25
60	132.3	180	312.3
70	154.35	245	399.35
80	176.4	320	496.4
90	198.45	405	603.45

54) Answers will vary.

55)

Price Reduction (dollars)	Price (dollars)	Number of Can Openers Sold	Calculation	Gross Profit (dollars)
0	24	0	$24 \cdot 0$	0
1	23	10	$23 \cdot 10$	230
2	22	20	$22 \cdot 20$	440
3	21	30	$21 \cdot 30$	630
4	20	40	$20 \cdot 40$	800
5	19	50	$19 \cdot 50$	950
6	18	60	$18 \cdot 60$	1080
7	17	70	$17 \cdot 70$	1190
8	16	80	$16 \cdot 80$	1280
9	15	90	$15 \cdot 90$	1350
10	14	100	$14 \cdot 100$	1400
11	13	110	$13 \cdot 110$	1430
12	12	120	$12 \cdot 120$	1440
13	11	130	$11 \cdot 130$	1430
14	10	140	$10 \cdot 140$	1400

56) Answers will vary.

57) 50
58) 950
59) Only up to a certain point. When they reduce the price more than 12 dollars, the gross profit starts to fall.
60) I would recommend that they reduce the price by $12, since that would provide a gross profit of $1440, the maximum amount possible.
61) C
62) B
63) D
64) A

Introduction to Exponential Functions

1)

Equation: $y = 7x$							
x	1	2	3	4	5	6	7
y	7	14	21	28	35	42	49

Equation: $y = x^2 + x$							
x	1	2	3	4	5	6	7
y	2	6	12	20	30	42	56

Continue the pattern.	Describe the pattern.
2) 1, 2, 4, 8, 16, 32, 64, 128, …	Multiply by 2
3) 1, 3, 9, 27, 81, 243, 729, 2187, …	Multiply by 3
4) 16, 8, 4, 2, 1, ½, ¼, ⅛, …	Multiply by ½ (or divide by 2)

5) You might notice that the outputs of linear and quadratic functions grow by adding quantities. This is the rate of change. Exponential functions grow by multiplying by a number.

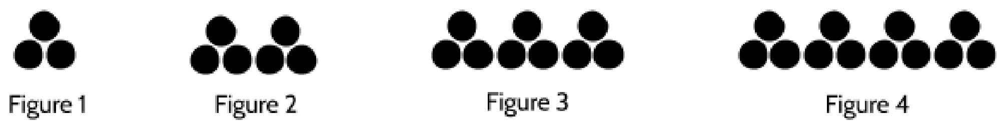
6) 1, 2, 4, 8, 16, 32, 64, 128, 256, 512

7) 64

8) 64

9) 4

10) Visual Pattern F



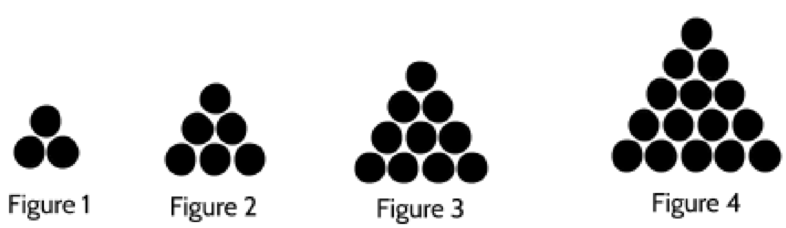
a)

Figure	Number of Circles
1	3
2	6
3	9
4	12
5	15
6	18

b)

c) Answers will vary.

11) Visual Pattern G



a)

b)

Figure	Number of Circles
1	3
2	6
3	10
4	15
5	21
6	28

c) Answers will vary.

12) Visual Pattern H

a) We should a few possibilities below, including the way we drew the 4th figure.

b)

Figure	Number of Circles
1	3
2	9
3	27
4	81
5	243
6	729

c) Answers will vary.

13) 81 circles

14) There should be 3 sections of the large figure.

15)

Fig. No.	1	2	3	4	5	6	7
No. of Circles	3	9	27	81	243	729	2187

16) Visual Pattern F is a <u>linear</u> growth pattern because the rate of change is constant.

17) If the function equation for Visual Pattern G includes n^2, it must be a quadratic growth pattern.

18) Visual Pattern H is an example of an exponential growth pattern because it grows by multiplying by a factor of 3 to create each new figure.

19) Visual Pattern G and Visual Pattern H are both nonlinear growth patterns because their rates of change are not constant.

20) H

21) F

22) G

23) This should be an exponential function. Can you show why that's true with drawings, a table, graph, or equation?

Three Views of an Exponential Function

1)

In	*Out*
0	0
1	2
2	4
3	6
4	8
5	10

Added +2 to each output.

2)

In	*Out*
0	1
1	2
2	4
3	8
4	16
5	32

Multiplied each output by 2.

3) The first function is growing by adding 2. The second function is growing by multiplying by 2.

4)

In	*Out*
0	1
1	4
2	16
3	64
4	256
5	1024

× 4, × 4, × 4, × 4, × 4

In	*Out*
0	1
1	5
2	25
3	125
4	625
5	3125

× 5, × 5, × 5, × 5, × 5

In	0	1	2	3	4	5
Out	5	10	20	40	80	160

× 2, × 2, × 2, × 2, × 2

5)

6) All the equations are equal to 2.

7) Yes. Each number is multiplied by 3 to get the next number.

8) The quotient of each equation is 3.

9) There is a constant growth factor of 4. The function is exponential.

10) There is not a constant growth factor. The function is not exponential.

11) There is a constant growth factor of 1.5. The function is exponential.

12) There is a constant factor of .5. The function is exponential. However, it is not a growth factor, but a decay factor, since the outputs are getting smaller.

13) 10, 11, 12.1, 13.31, 14.641. This is growth, since the number is getting bigger. It is exponential because it is changing through multiplication.

10, 9, 8.1, 7.29, 6.561. This is decay, since the number is getting smaller. It is exponential because it is changing through multiplication.

14) The outputs would decay or get smaller.

15) The outputs would grow or get larger.

16) 1, 5, 25, 125, 625, 3125.

17) 625

18) 15,625

19) 2

20) 243

21)

Equation: $y = 3^x$			
x	Exponential Form	Factors	y
0	3^0	1	1
1	3^1	$1 \cdot 3$	3
2	3^2	$1 \cdot 3 \cdot 3$	9
3	3^3	$1 \cdot 3 \cdot 3 \cdot 3$	27
4	3^4	$1 \cdot 3 \cdot 3 \cdot 3 \cdot 3$	81

22)

Equation: $y = (2.5)^x$			
x	Exponential Form	Factors	y
4	$(2.5)^4$	$1 \cdot 2.5 \cdot 2.5 \cdot 2.5 \cdot 2.5$	39.0625
2	$(2.5)^2$	$1 \cdot 2.5 \cdot 2.5$	6.25
0	$(2.5)^0$	1	1
6	$(2.5)^6$	$1 \cdot 2.5 \cdot 2.5 \cdot 2.5 \cdot 2.5 \cdot 2.5 \cdot 2.5$	244.140625
3	$(2.5)^6$	$1 \cdot 2.5 \cdot 2.5 \cdot 2.5$	15.625

23)

Equation: $y = 10 \cdot 2^x$			
x	Exponential Form	Factors	y
0	$10 \cdot 2^0$	10	10
1	$10 \cdot 2^1$	$10 \cdot 2$	20
2	$10 \cdot 2^2$	$10 \cdot 2 \cdot 2$	40
3	$10 \cdot 2^3$	$10 \cdot 2 \cdot 2 \cdot 2$	80

24)

Equation: $y = 100 \cdot 2^x$			
x	Exponential Form	Factors	y
0	$100 \cdot 2^0$	100	100
1	$100 \cdot 2^1$	$100 \cdot 2$	200
2	$100 \cdot 2^2$	$100 \cdot 2 \cdot 2$	400
3	$100 \cdot 2^3$	$100 \cdot 2 \cdot 2 \cdot 2$	800

25) You might notice that the outputs in the last two tables are very similar. The starting amount times the power creates the output. Without the starting amount of 10 and 100, the outputs of 2^x would be 1, 2, 4, 8, etc. The outputs with the starting amount are the product of the starting amount and 2^x.

26) 450

27) 12.5

28) $2,208

29) 28,925

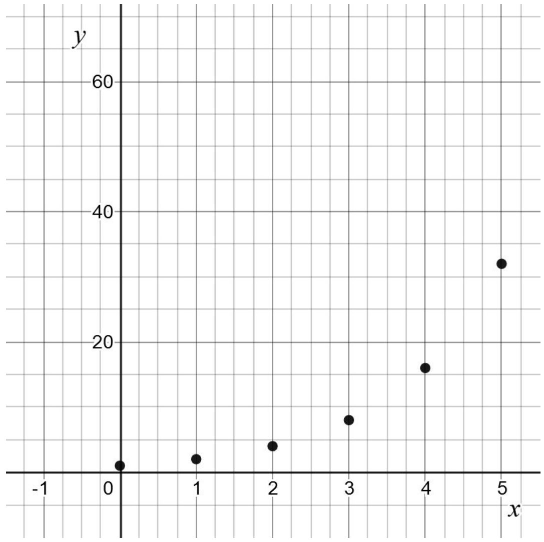
30) Growth, since the factor of 1.4 is more than 1.

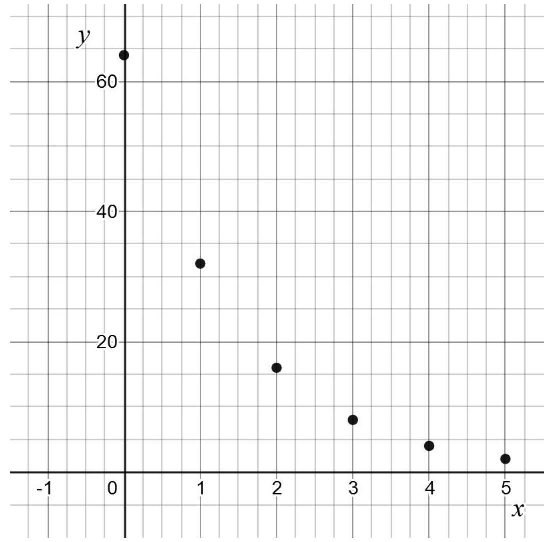
31)

Equation: $y = 1 \cdot 2^x$	
x	y
0	1
1	2
2	4
3	8
4	16
5	32

Equation: $y = 64 \cdot (.5)^x$	
x	y
0	64
1	32
2	16
3	8
4	4
5	2

32)





33) Answers will vary.

34) The numbers (1, 2, 4, 8, 16, 32, 64, etc.) are really small compared to the size of the numbers as they grow exponentially. For example, the y value for $x = 20$ is more than 1,000,000!

35) The missing y value is .25

36) Surprisingly, it will never go below the x axis. The outputs will get smaller and smaller, but will never disappear entirely. Half of a half of a half of a half of a ...

37) Graph I. It might grow slowly at first, but eventually it will grow very quickly. This happens will all exponential functions with a factor larger than 1.

38)

x	-3	-2	-1	0	1	2	3	4	5
$2x$	−6	−4	−2	0	2	4	6	8	10
x^2	9	4	1	0	1	4	9	16	25

This number sentence is sometimes true. x^2 is greater than $2x$ when $x > 2$. For example, if $x = 3$, then $x^2 = 9$ and $2x = 6$. If $x = 2$, then $x^2 = 4$ and $2x = 4$ as well. If $x = 1$, $2x$ is greater than x^2, since $2(1) = 2$ and $1^2 = 1$.

39) This number sentence is sometimes true. Can you explain why it isn't always true?

40) This number sentence is true for two inputs for x. Can you find the two inputs?

41) This number sentence is true for many numbers, but there are some inputs that make it false. Can you find them?

42) This is almost always true. There are two inputs that makes this number sentence false.

43) This statement is always true. Why does x have to be greater than 0 for this statement to be true?

Exponential Functions in the World

1) Answers will vary.

2) $f(n) = 1000n$

3) $30,000

4) $1,000-per-day method since you would get $10,000. You would only get $10.24 on the final day of the doubling-pennies method.

5) The doubling-pennies method. You would get $167,772.16 on the final day of work, compared with a total amount of $25,000 with the other method.

6) On the 22nd day, with the doubling-pennies method, you would earn $20,971.52. If you add up all the previous day's salary, you would make a total of $41,943.03. This is a lot more than the total of $22,000 you would make with the $1,000-per-day method. If you work fewer days, the $1000-per-day method is better.

7) Answers will vary.

8) 25

9)

Day #	Calculation	Number of people receiving post this week
1	$1 \cdot 5$	5
2	$1 \cdot 5 \cdot 5$	25
3	$1 \cdot 5 \cdot 5 \cdot 5$	125
4	$1 \cdot 5 \cdot 5 \cdot 5 \cdot 5$	625
5	$1 \cdot 5 \cdot 5 \cdot 5 \cdot 5 \cdot 5$	3,125
6	$1 \cdot 5 \cdot 5 \cdot 5 \cdot 5 \cdot 5 \cdot 5$	15,625
7	$1 \cdot 5 \cdot 5 \cdot 5 \cdot 5 \cdot 5 \cdot 5 \cdot 5$	78,125
8	$1 \cdot 5 \cdot 5 \cdot 5 \cdot 5 \cdot 5 \cdot 5 \cdot 5 \cdot 5$	390,625
9	$1 \cdot 5 \cdot 5 \cdot 5 \cdot 5 \cdot 5 \cdot 5 \cdot 5 \cdot 5 \cdot 5$	1,953,125
10	$1 \cdot 5 \cdot 5 \cdot 5 \cdot 5 \cdot 5 \cdot 5 \cdot 5 \cdot 5 \cdot 5 \cdot 5$	9,765,625

10) 13 days. $5^{12} =$ 244,140,625 (244 million people). $5^{13} =$ 1,220,703,125 (1.2 billion people).

11) C

12) C

13) Change to 6^n.

14) Answers may vary.

15) Answers may vary.

16) 8

17) 32

18) You will know the answer by the end of this activity.

19)

Hours	20-minute increments
1	3
2	6
3	9
4	12
h	15

20)

Time	Number of 20-min increments (n)	Number of bacteria	Number of bacteria (exponential form)
0:00:00	0	$1 = 1$	2^0
0:20:00	1	$2 = 1 \cdot 2$	2^1
0:40:00	2	$4 = 1 \cdot 2 \cdot 2$	2^2
1:00:00	3	$8 = 1 \cdot 2 \cdot 2 \cdot 2$	2^3
1:20:00	4	$16 = 1 \cdot 2 \cdot 2 \cdot 2 \cdot 2$	2^4
1:40:00	5	$32 = 1 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2$	2^5
2:00:00	6	$64 = 1 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2$	2^6
2:20:00	7	$128 = 1 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2$	2^7
2:40:00	8	$256 = 1 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2$	2^8
3:00:00	9	$512 = 1 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2$	2^9
	n		2^n

21) 4,096 bacteria: $2^{12} = 4,096$

22) 6 hours and 40 minutes (twenty 20-minute increments): $2^{20} = 1,048,576\ bacteria$

23) 4.7×10^{21} (4.7 sextillion or 4.7 and twenty zeros): 2^{72} is approximately equal to $4,700,000,000,000,000,000,000$

24) Answers may vary.

25) Answers may vary.

26)

Day	How many people are told the rumor during the day?	How many people know the rumor by the end of the day?
1	1	2
2	2	4
3	4	8
4	8	16
5	16	32
6	32	64

27) 6 days

28) 7 days

29)

Original Amount		Growth Rate		Growth Factor	
Percentage	Decimal	Percentage	Decimal	Percentage	Decimal
100%	1	100%	1	200%	2
100%	1	90%	0.9	190%	1.9
100%	1	80%	0.8	180%	1.8
100%	1	45%	0.45	145%	1.45
100%	1	20%	0.2	120%	1.2
100%	1	5%	0.05	105%	1.05
100%	1	1%	0.01	101%	1.01

30)

Day	Multiplying by Growth Factor	Exponent	Total Number of "Infections"
1	$1 \cdot 2$	2^1	2
2	$1 \cdot 2 \cdot 2$	2^2	4
3	$1 \cdot 2 \cdot 2 \cdot 2$	2^3	8
4	$1 \cdot 2 \cdot 2 \cdot 2 \cdot 2$	2^4	16
5	$1 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2$	2^5	32
n		2^n	

31) C

32) B

33)

Day	Multiplying by Growth Rate	Exponent	Total Number of Infections (Round to whole number)
1	$1 \cdot 1.9$	1.9^1	1.9 → 2
2	$1 \cdot 1.9 \cdot 1.9$	1.9^2	3.61 → 4
3	$1 \cdot 1.9 \cdot 1.9 \cdot 1.9$	1.9^3	6.85... → 7
4	$1 \cdot 1.9 \cdot 1.9 \cdot 1.9 \cdot 1.9$	1.9^4	13.03...→ 13
5	$1 \cdot 1.9 \cdot 1.9 \cdot 1.9 \cdot 1.9 \cdot 1.9$	1.9^5	24.76... → 25
n		1.9^n	

34) 25

35) 170

36) A

37) 613

38) D

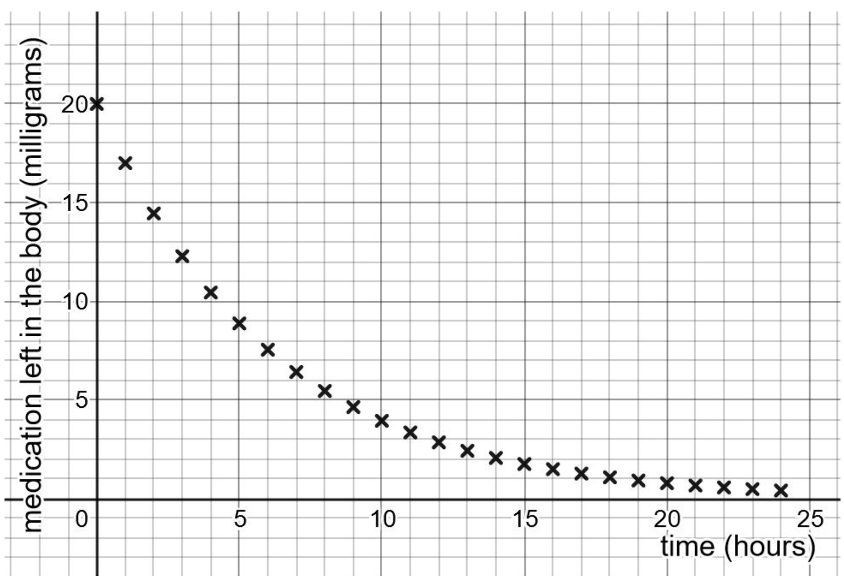
39) A. $1.5^{18} = 1478$ people

40)

Hour t	Formula $20 \cdot (0.85)^t$	Calculation	Medication left (mg) (Round to tenths place) $f(t)$
0	$20 \cdot (0.85)^0$	20 · 1	20
1	$20 \cdot (0.85)^1$	20 · 1 · 0.85	17
2	$20 \cdot (0.85)^2$	20 · 1 · 0.85 · 0.85	14.45 → 14.5
3	$20 \cdot (0.85)^3$	20 · 1 · 0.85 · 0.85 · 0.85	12.28… → 12.3
4	$20 \cdot (0.85)^4$	20 · 1 · 0.85 · 0.85 · 0.85 · 0.85	10.44… → 10.4
5	$20 \cdot (0.85)^5$	20 · 1 · 0.85 · 0.85 · 0.85 · 0.85 · 0.85	8.87…→ 8.9

41) 5.4 milligrams

42)



Complete the table using the function $f(t) = 20 \cdot (.85)^t$, then plot the points on the graph. Some have been done for you.

t	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24
$f(t)$	20.0	17.0	14.5	12.3	10.4	8.9	7.5	6.4	5.4	4.6	3.9	3.3	2.8	2.4	2.1	1.7	1.5	1.3	1.1	.9	.8	.7	.6	.5	.4

43) Answers may vary.

44) Decay.

45) C

Test Practice Questions

1) C
2) Part I: B, Part II: D
3) D
4) C
5) Part I: C, Part II: C
6) D
7) C
8) D
9) D
10) 5
11) B
12) C
13) C
14) A
15) B
16) D
17) A
18) C
19) D
20) C
21) C
22) D
23) Part I: A, Part II: C
24) B
25) D
26) B
27) B
28) A
29) C
30) D
31) Part I: C, Part II: A
32) B
33) A
34) Part I: D, Part II: C

Vocabulary Review

You can use this section to look up words used in this math packet.

consecutive (adjective): Numbers which follow each other one after the other, without gaps. The numbers 3, 4, 5, 6 are in consecutive order.

constant (noun): A value in an expression or equation that doesn't change. For example, in the function $y = 4x + 6$ the numbers 4 and 6 are both constants.

constant (adjective): Something that stays the same. A *constant* rate of growth means that the rate stays the same over time.

distributive property of multiplication (noun): a mathematical property that shows multiplying a number by a group of numbers added together is the same as doing each multiplication separately. For example, $3 \cdot (2 + 4)$ is equal to $3 \cdot 2 + 3 \cdot 4$.

equal sign: a symbol used to show symmetric balance between two values or quantities, one on each side of the equal sign. Can be read as "is equivalent to" or "is the same as".

equation (noun): A number sentence that shows two expressions are equal by using the equal sign. $2^3 = 8$ is an *equation*. $5x + 3$ is an expression, <u>not</u> an *equation*.

equivalent (adjective): Having the same value. For example, 4 quarters and 20 nickels are equivalent. Eight hours is equivalent to 28,800 seconds. 4^3 and 64 are equivalent.

estimate (verb): To make a rough guess at a number, usually without making written calculations

evaluate (verb): To calculate the value of something. If asked to *evaluate* 4^3, your answer should be 64.

expression (noun): Numbers and symbols that show the value of something. 100, $5x + 3$, and 2^3 are all *expressions*. $5x + 3 = 23$ is an equation made up of two *expressions*.

exponent (noun): In a quantity represented as a power, the *exponent* shows how many times the base is multiplied. The exponent is shown as a smaller number up and to the right of the base. For example, in the power 2^3, the *exponent* is 3.

exponential growth (adjective): A growth pattern related to multiplying by a repeated factor. $y = 2^x$ is a common example of an exponential function where 2 is used as a factor x times to calculate the output of y.

factor (noun): Whole numbers that are multiplied together to get another number. A number that can be divided into another number evenly, with no remainder.

factor (verb): To split a number into its factors (see above definition of factors).

function (noun): a relationship between two quantities. For every input into a function, there can only be one output. A function can be represented by an equation, a table, a graph, and a verbal or written description.

generalize (verb): To look at specific examples and realize that something is true *in general*. For example, what happens with you divide a number by itself? Specific examples: $8 \div 8 = 1$ or $25 \div 25 = 1$ *or* $.75 \div .75 = 1$. Example of a *generalization*: Any number divided by itself is 1.

graph (noun): A visual representation of a function relationship.

input (noun): The number that goes into a function.

linear function (noun): A function that has a constant rate of change. The graph of a linear function is a straight line.

multiple (noun): A number that can be divided by another number evenly, with no remainder. 25 is a *multiple* of 5.

nonlinear function (noun): A function that does not have a constant rate of change. The graph of a nonlinear function is not a straight line. There are many kinds of nonlinear functions. Two of them are *quadratic* functions and *exponential* functions.

ordered pair (noun): Ordered pairs are used to show a position on a graph. An ordered pair is made up of an input and an output that make a function true. They are usually written in parentheses like this (14, 7), where the first number shows the x value (horizontal) and the second number shows the y value (vertical).

output (noun): The value that comes out of a function when each input goes in. The output is dependent on the input.

parabola (noun): The shape of the graph of a quadratic function.

pattern (noun): Patterns are things we can observe. We gather information about what has come before to make predictions about what will come next.

property (noun): A character or quality that something has. In science, we use the physical property to refer to color, texture, density, and other qualities of physical objects. In mathematics, properties refer to characteristics of numbers or operations. Example:

> *Commutative property*: Addition and multiplication are *commutative*, meaning the order of the operation doesn't matter. $2+7=7+2$ and $2 \cdot 7 = 7 \cdot 2$

product (noun): The result of multiplication. 4 times 5 gives a *product* of 20.

quadratic growth (noun): A growth pattern related to the size of different squares. A quadratic function in standard form looks like $y = ax^2 + bx + c$ where a can be any number except 0. The x^2 (x "squared") term in the equation makes the function quadratic.

rate of change (noun): The rate of change in the outputs when the inputs are consecutive. Known as *slope* when a part of the graph of an equation.

root (noun): A *root* is a value you can put into a function as an input for x so that 0 is an output for y. For example, the x-values -3 and -2 are the roots of the function $y=(x+2)(x-3)$. In a graph, the x-intercepts (points where a line intersects with the x axis) are the same as the *roots.* The word *zero* is sometimes used to mean the same thing as *root*. The roots, x-intercepts, and zeros (all which mean the same thing) are the solutions to a function equation.

> *square root*: A square root of a number is a value that, when multiplied by itself, gives the number. The *square root* of 25 is 5.

starting amount (noun): The value of the output when the input is zero. Known as y-*intercept* when a part of the graph of an equation.

term (noun): A single number or variable, or numbers and variables multiplied together. Terms are separated by + or - signs. For example, there are three terms in the equation $2+5=7$.

variable (noun): A letter or symbol that represents another value, either any number, a specific number, or a set of numbers. In the expression x^3, *x is a variable* that could mean any number. Variables can also represent other things. For example, in geometry, points and angles are represented by letters.

vertex (noun): The highest or lowest point on a parabola.

x-axis: The line that goes from left to right in a graph. The horizontal axis on the coordinate grid.

x-intercept (noun): The point of intersection with the x-axis of a graph.

y-axis (noun)**:** The line that goes from top to bottom in a graph. The vertical axis on the coordinate grid.

y-intercept (noun): The point of intersection with the y-axis of a graph.

zero-product property (noun): This fact has to do with multiplication. If two numbers multiplied together gives a result of 0, then one of the two numbers must equal zero. It is also written this way: If $a \cdot b = 0$, then a or b has to be equal to 0.

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