

# Tools of Algebra: Nonlinear Functions

*Fast Track GRASP Math Packet*

Part 1

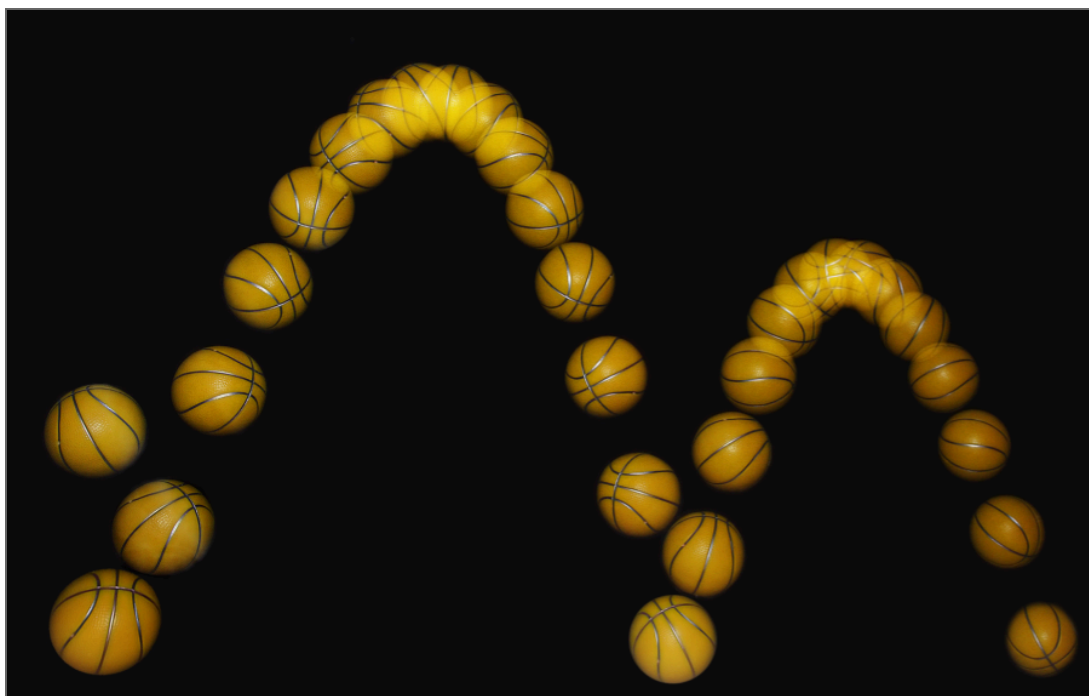


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Version 1.6

Released 1/22/2025



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## Overview

In this packet you will explore nonlinear functions, which are functions that represent nonlinear growth and result in graphs that are not straight lines.

<b>Prerequisites</b>	Before working on this packet, you should complete the following packets: <ul style="list-style-type: none"><li>● <i>Number Lines to Coordinate Planes</i> Fast Track GRASP Math Packet, or feel confident locating and plotting points on the coordinate plane.</li><li>● <i>Tools of Algebra: Linear Functions</i> Fast Track GRASP Math Packet</li><li>● <i>The Power of Exponents</i> Fast Track GRASP Math Packet</li></ul>
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In Part 1, you will study the following topics:

- Comparing linear and nonlinear growth
- Area models and quadratic functions
- Three views of a quadratic functions: tables, equations, and graphs
- Exploring quadratic growth in visual patterns

In Part 2, you will build on what you learned in Part 1, and study the following topics:

- Exploring quadratic growth in real-world situations
- Exponential growth in number patterns and visual patterns\*
- Three views of exponential functions: tables, equations, and graphs\*
- Exploring exponential growth in real-world situations\*

In addition to the learning the topics above, you will find the following materials to help you:

- A review of the big ideas at the end of Part 1 and Part 2.
- High School Equivalency Test Practice Questions. You will practice all the concepts you have learned from this packet (parts 1 and 2) to work on these questions.
- A graphic organizer to help you study vocabulary and a glossary with important terms from this packet for you to study.
- Concept Circles can help you make connections between the concepts you have learned and to help you remember those connections.

*\* Exponents and scientific notations are part of the GED exam. While exponential functions are an important math topic (especially in science), you will probably not see them on the test.*

## Assessment Questions

*Calculator allowed*

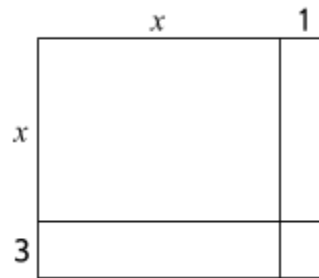
The following questions will help to see if this packet is right for you. Do your best to answer each question below. When you are finished with the questions, read our recommendations.

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### Question 1

Which expression shows the area of the rectangle below?

- A.  $x^2 + 3x + 4$
- B.  $x^2 + 3x$
- C.  $x^2 + 4x + 3$
- D.  $3x^2 + x$



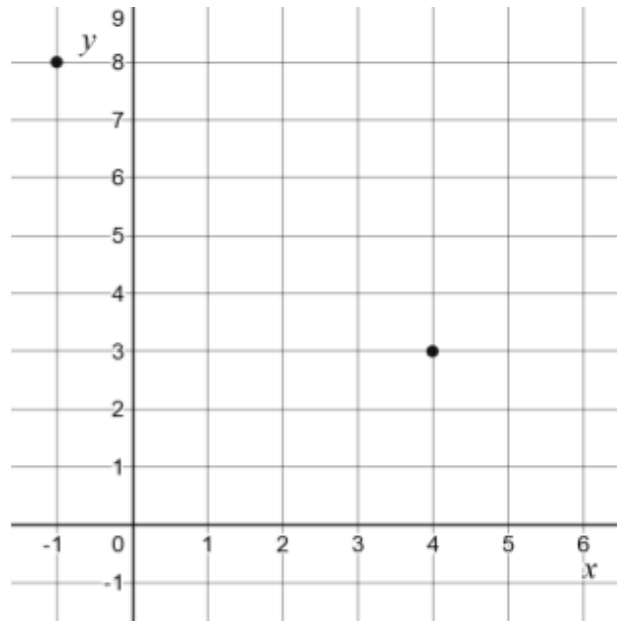
Tools of Algebra: Nonlinear Functions (Part 1)

Question 2

Complete the function table for  $y = x^2 - 4x + 3$  and plot the rest of the points on the graph. Two points have been plotted for you.

*Note: The ordered pair shows the location of a point on a graph. The first number is the x value and the second number is the y value.*

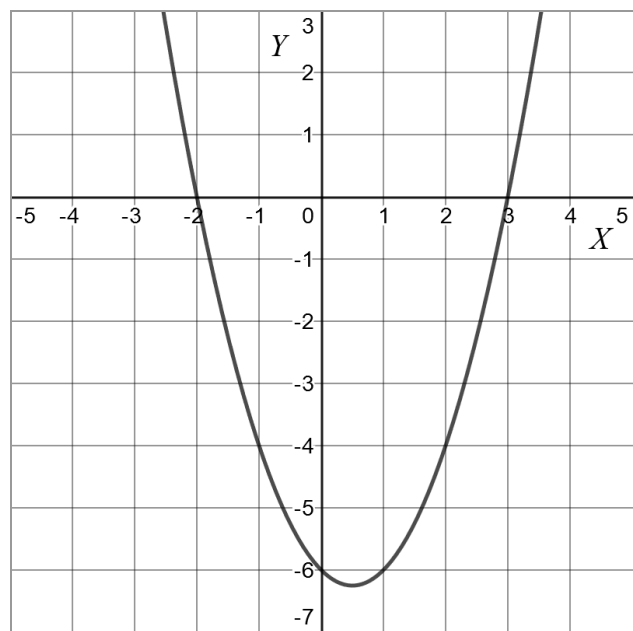
Equation: $y = x^2 - 4x + 3$		
$x$	$y$	Ordered Pair
-1	8	(-1, 8)
0		
1		
2	-1	
3		
4	3	(4, 3)
5		



Question 3

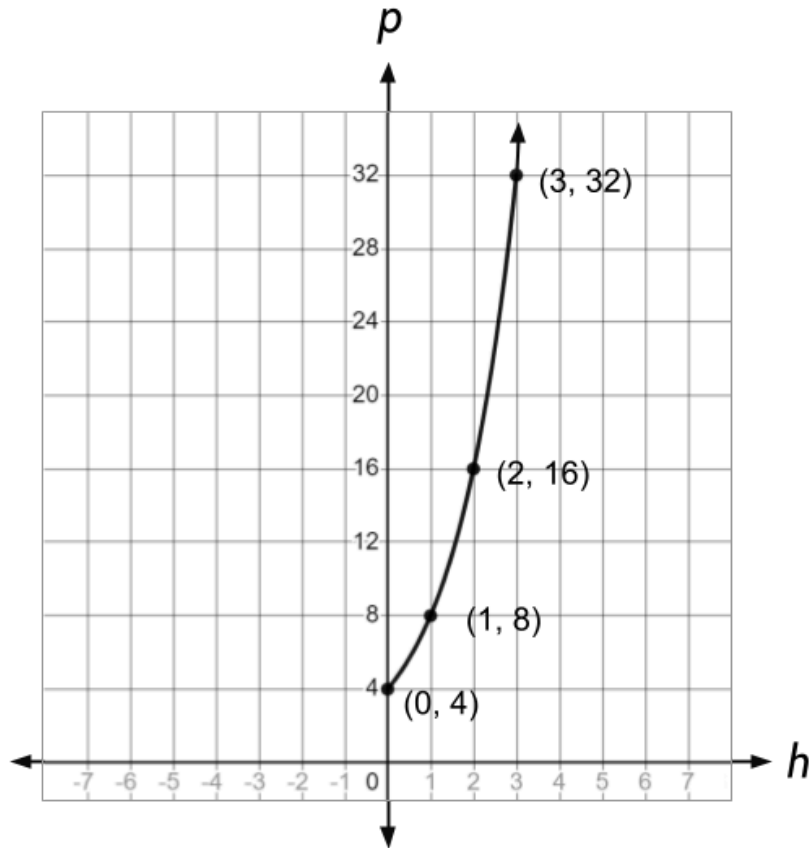
What are the roots of the quadratic equation associated with the graph?

- A. -6 and 3
- B. -6 and 0
- C. -3 and 2
- D. -2 and 3



Question 4

Liz collected population data,  $p$ , from a colony of E. coli bacteria over time in hours,  $h$ , as shown in the graph.



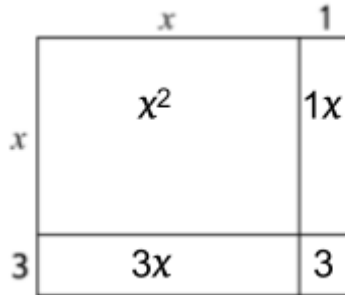
Which equation matches the data in the graph?

- A.  $p = 4h + 4$
- B.  $p = 4h^2$
- C.  $p = 4 \cdot 2^h$
- D.  $p = 2 \cdot 4^h$

## Answer Key

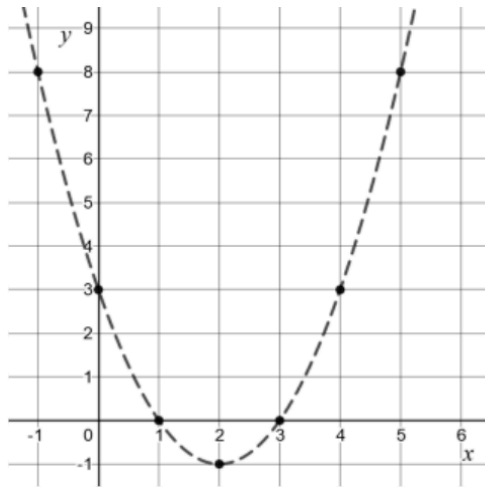
### Question 1

Choice C.  $x^2 + 4x + 3$



### Question 2

Equation: $y = x^2 - 4x + 3$		
$x$	$y$	Ordered Pair
-1	8	(-1, 8)
0	3	(0, 3)
1	0	(1, 0)
2	-1	(2, -1)
3	0	(3, 0)
4	3	(4, 3)
5	8	(5, 8)



### Question 3

Choice D. -2 and 3

### Question 4

Choice C.

Hours ( $h$ )	$4 \cdot 2^h$	Bacteria ( $p$ )
0	$4 \cdot 2^0$	4
1	$4 \cdot 2^1$	8
2	$4 \cdot 2^2$	16
3	$4 \cdot 2^3$	32

## Recommendations

Consider the following when making a decision about working through this packet:

- Student has some difficulty with Question 1 or 2: The student may choose to work through the whole packet, both Part 1 and Part 2.
- Student has some difficulty with Question 3 or 4: If a student comfortably answers Questions 1 & 2, but has some difficulty with Question 3 or 4, the student may feel confident enough to skip Part 1 of the packet and go directly to Part 2.
- Student comfortably answers all four questions: The student may choose to work on a different packet. However, it is recommended that students complete the Test Practice Questions in Tools of Algebra: Nonlinear Functions, Part 2, for additional practice before they take the GED exam.

This assessment asks students to demonstrate understanding of:

Question 1: (*from Tools of Algebra: Nonlinear Functions, Part 1*): Multiplying polynomials using an area model (GED Algebraic Problem Solving Assessment Targets Content Indicators: A.4.a)

Question 2 (*from Tools of Algebra: Nonlinear Functions, Part 1*): Function notation, calculating outputs of a function, locating points on the coordinate grid (GED Algebraic Problem Solving Assessment Targets Content Indicators: A.5.a and A.7.b)

Question 3 (*from Tools of Algebra: Nonlinear Functions, Part 1*): Key features of linear, quadratic, and exponential graphs (GED Algebraic Problem Solving Assessment Targets Content Indicators: A.5.e)

Question 4: (*from Tools of Algebra: Nonlinear Functions, Part 1*): The relationship between an exponential function and the graph that represents the function (GED Algebraic Problem Solving Assessment Targets Content Indicators: A.1.i, A.4.a, A.5.a)



## Welcome!

Congratulations on deciding to continue your learning! We are happy to share this study packet on non-linear functions, seen through rules, tables, graphs, and equations. We hope that these materials are helpful in your efforts to earn your high school equivalency diploma. This group of math study packets will cover mathematics topics that we see on high school equivalency exams. If you study these topics carefully, while also practicing other math skills, you will increase your chances of passing the exam.

Please take your time as you go through the packet. You will find plenty of practice here, but it's useful to make extra notes for yourself to help you remember. You will probably want to have a separate notebook where you can recopy problems, write questions and include information that you want to remember. Writing is thinking and will help you learn.

After each section, you will find an answer key. Try to answer all the questions and then look at the answer key. It's not cheating to look at the answer key, but do your best on your own first. If you find that you got the right answer, congratulations! If you didn't, it's okay. This is how we learn. Look back and try to understand the reason for the answer. Please read the answer key even if you feel confident. We added some extra explanations and examples that may be helpful. If you see a word that you don't understand, try looking at the *Vocabulary Review* at the end of the packet. There are some strategies for learning new words described in the next few pages.

We hope you share what you learn with your friends and family. If you find something interesting here, tell someone about it! If you find a section challenging, look for support. If you are in a class, talk to your teacher and your classmates. If you are studying on your own, talk to people you know or try searching for a phrase online. Your local library should have information about adult education classes or other support. You can also find classes listed here: <http://www.acces.nysed.gov/hse/hse-prep-programs-maps>

You are doing a wonderful thing by investing in your own education right now. You have our utmost respect for continuing to learn as an adult.

Please feel free to contact us with questions or suggestions.

Best of luck!

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CUNY Adult Literacy and High School Equivalency Program

## Vocabulary

It is important to understand mathematical words when you are learning new topics. The following vocabulary will be used a lot in this study packet:

constant · curve · exponential · function · line · linear · nonlinear · quadratic

In this first activity, you will think about each word and decide how familiar you are with it. For example, think about the word “cube.” Which of these statements is true for you?

- I know the word “cube” and use it in conversation or writing.
- I know the word “cube,” but I don’t use it.
- I have heard the word “cube,” but I’m not sure what it means.
- I have never heard the word “cube” at all.

In the chart on the next page, read each word and then choose one of the four categories and mark your answer with a ✓ (checkmark). Then write your best guess at the meaning of the word in the right column. If it’s easier, you can also just use the word in a sentence.

Here’s an example of how the row for “area” might look when you’re done:

Word	I know the word and use the word	I know the word but don’t use it	I have heard the word, but I’m not sure what it means	I have never heard the word	My best guess at the meaning of the word (or use the word in a sentence)
cube	✓				like a box, all the sides are the same length

Complete the table on the next page.

Tools of Algebra: Nonlinear Functions (Part 1)

Word	I know the word and use the word	I know the word but don't use it	I have heard the word, but I'm not sure what it means	I have never heard the word	My best guess at the meaning of the word (or use the word in a sentence)
<b>constant</b>					
<b>curve</b>					
<b>exponential</b>					
<b>function</b>					
<b>line</b>					
<b>linear</b>					
<b>nonlinear</b>					
<b>quadratic</b>					

## Introduction to Nonlinear Functions

### Linear vs. Nonlinear Growth

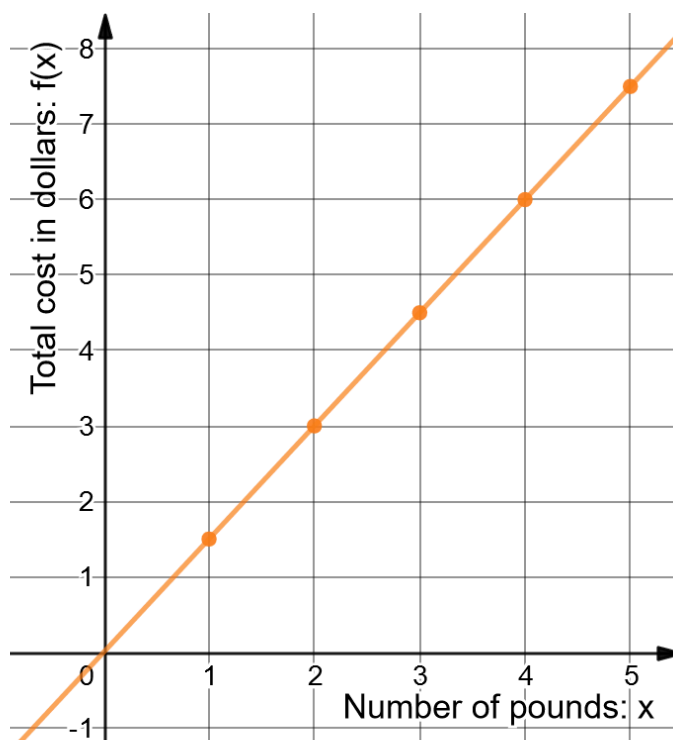
Functions are a way to understand how different quantities grow in relation to other quantities. Some quantities grow in a linear way. This means that the same amount is added on while the quantity grows. For example, the price of different amounts of oranges grows in a linear way. If 1 pound of oranges costs \$1.50, then 2 pounds costs \$3.00, 3 pounds cost \$4.50, and so on. For each additional pound, another \$1.50 is added on.

A function rule for the price of oranges might look like this:  $f(x) = 1.5x$ . The variable  $x$  represents the number of pounds of oranges and  $f(x)$  represents the total cost of  $x$  pounds of oranges.

We could also graph the price of oranges (shown on the right). Notice how a graph of the points makes a straight line? That's why  $f(x) = 1.5x$  is called a “**linear**” function.

Another way to write the function is with a linear equation,  $y = 1.5x$ . Both linear functions and linear equations make lines when inputs and outputs are plotted on a graph.<sup>1</sup>

- 1) How much would 4 pounds of oranges cost?



<sup>1</sup> You can find practice with linear functions in the **Tools of Algebra: Linear Functions** packets..

## Tools of Algebra: Nonlinear Functions (Part 1)

Another example of linear growth is the size of a paycheck based on an hourly wage. If you work for \$15.00/hr., the size of your paycheck before taxes is 15 multiplied by the number of hours. The rate of change of 15 would be constant, since you add \$15.00 to the paycheck for each additional hour of work..



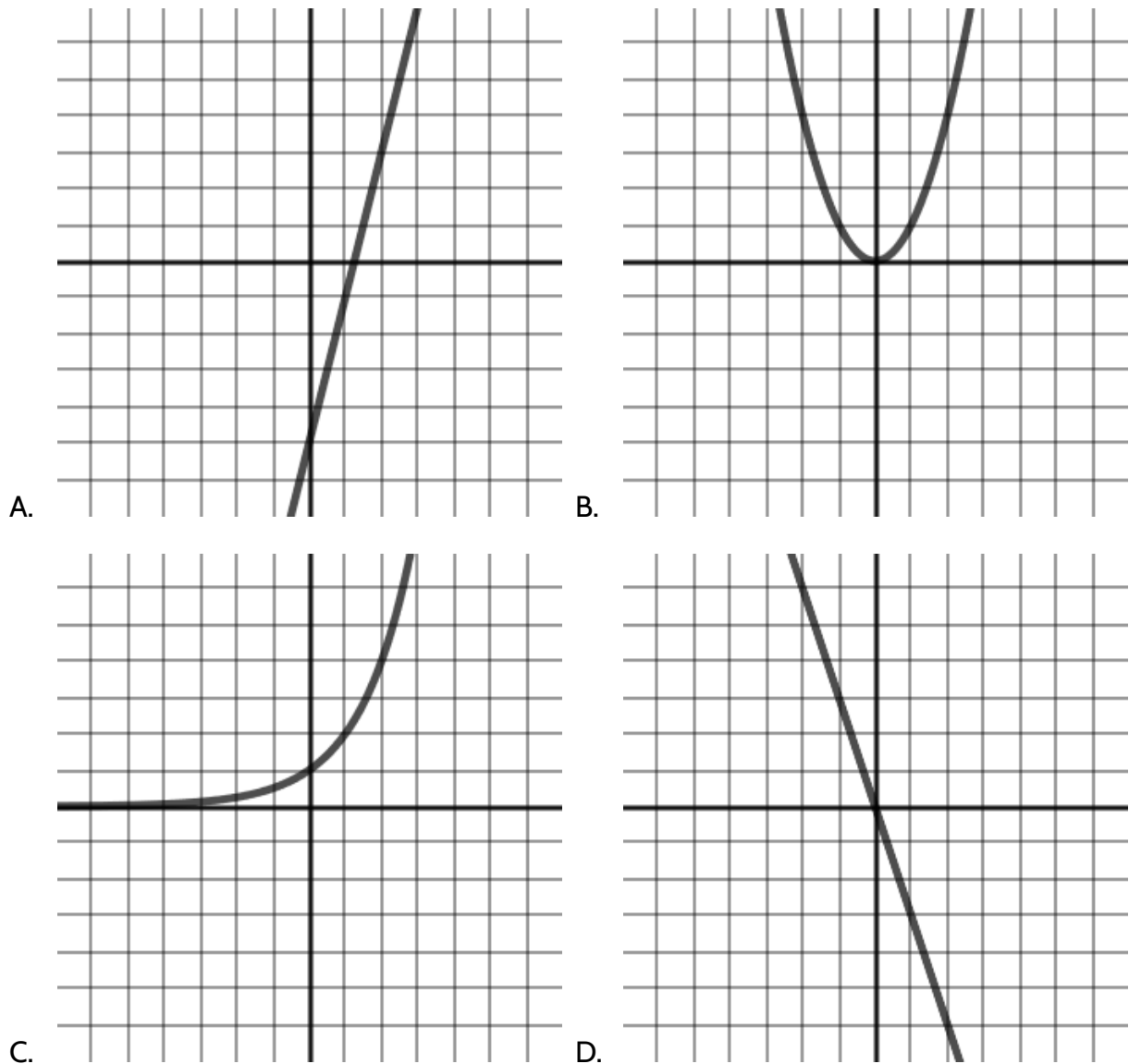
There are many examples of linear growth in the world: The price of produce per pound, the size of a paycheck with an hourly wage, the cost of a taxi ride, the amount of tax paid on a purchase, conversion of temperature in Fahrenheit to Celsius, etc. All of these situations involve functions where one quantity grows steadily in relationship to another quantity.

If you plotted data from any of the situations above, you would get a line graph. This is because the rate of change is constant. In order to predict the next consecutive amount, you can always add the rate of change.

**The rate of change is the change in the outputs when the inputs are consecutive.**

However, linear growth is not the only way that growth happens in the world. There are many situations where the amount of growth is not constant. In these situations, the growth is *nonlinear*, meaning it would not form a line on a graph if you plotted the data points. With nonlinear growth, there isn't a constant rate of change, so there isn't one number we can add to get the next number in a pattern. However, there are often other ways of using patterns to understand nonlinear growth.

2) Which of these graphs show nonlinear functions?

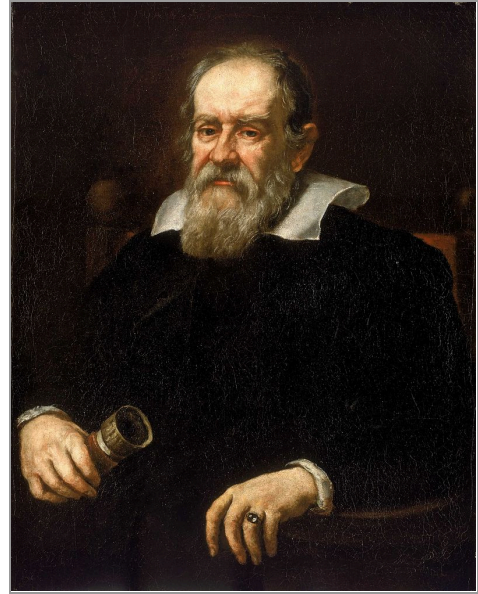


We will look at two kinds of nonlinear growth in this packet, *quadratic* functions and *exponential* functions. For each type of functions, we will look at number patterns, visual patterns, function tables, equations, and graphs. We will also look at examples of how quadratic and exponential functions are used to understand different kinds of growth in the world.

Note: We recommend using TI-30XS calculator throughout this packet.

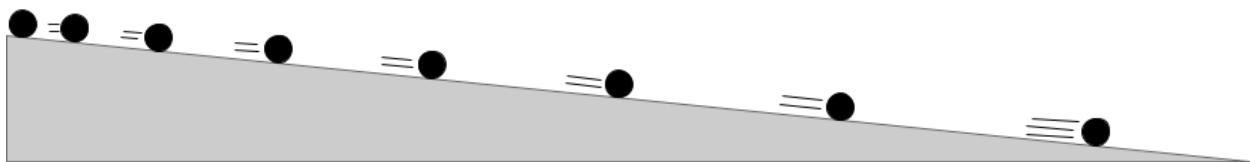
## Investigating Gravity

Galileo Galilei (1564-1642) was a scientist and mathematician who lived in the city of Pisa in what is now Italy. He was interested in the movement of planets, moons, and stars, and built telescopes to better observe them. Galileo was able to see that there are mountains on the moon, for example.



At the time, the Catholic Church taught that Earth was the center of the universe and did not move. They believed that all planets and stars revolved around Earth. Based on the work of Nicolaus Copernicus and his own observations, Galileo believed that the planets in the solar system, including Earth, revolved around the sun. Galileo was eventually arrested and found guilty of having beliefs in contradiction to the Church's teaching. He was sentenced to house arrest for the rest of his life and his books were banned. He experienced another tragedy as an old man when he became blind due to looking at the sun through a telescope.

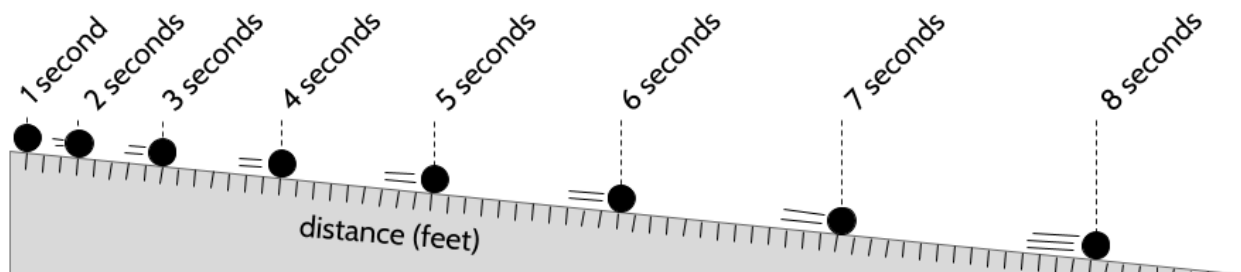
Even while confined to his home, Galileo continued to make many contributions to the field of science. One of Galileo's major discoveries was how Earth's gravity affects falling objects. It was really difficult to measure the speed of falling objects since they move so quickly. In order to understand gravity, Galileo figured out a way to reduce the effects of gravity and slow down falling objects so that he could see what was happening. In his experiment, he rolled balls down inclined planes to see how quickly the balls sped up. He placed each ball at the top of a ramp and then measured how far it went as time went by.



3) What do you notice about how the ball is moving down an inclined plane?

Tools of Algebra: Nonlinear Functions (Part 1)

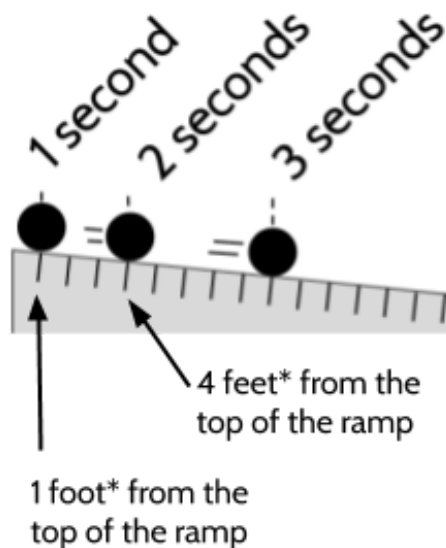
Galileo noticed a pattern in the distance that the ball traveled as it rolled down the ramp. The diagram below shows a modern version of the inclined plane experiment, using current technology to measure time and distance. Imagine that a photograph is taken every second as the ball rolls down the ramp. The horizontal marks below the ramp show the distance the ball travels.



After 1 second, the ball travels 1 foot.

- 4) How far does the ball travel after 2 seconds?
- 5) How far does the ball travel after 3 seconds?
- 6) Complete the table.

time (seconds)	distance (feet)
0	0
1	1
2	4
3	
4	
5	
6	
7	
8	



\*You can count the tick marks to see how many feet the ball has traveled.



*Tools of Algebra: Nonlinear Functions (Part 1)*

- 7) If the ramp was longer and the ball continued rolling, how far would it travel after 10 seconds?
- 8) Galileo looked at the distance traveled and noticed some patterns. What patterns do you notice in the distances traveled?

To find patterns in a function table, one thing we can do is look at the rate of change in the outputs. For example, from 0 seconds to 1 second, the change in distance is 1 foot. From 1 second to 2 seconds, the change in distance is 3 feet.

- 9) Write in the change in distance for the rest of the table on the right.

The distance traveled in a certain amount of time is the speed. The change in distance in each second is the speed of the ball as it rolls.

- 10) Is the ball rolling at a constant speed? How do you know?

time (seconds)	distance (feet)	change in distance
0	0	
1	1	+1
2	4	+3
3	9	
4	16	
5	25	
6	36	
7	49	
8	64	

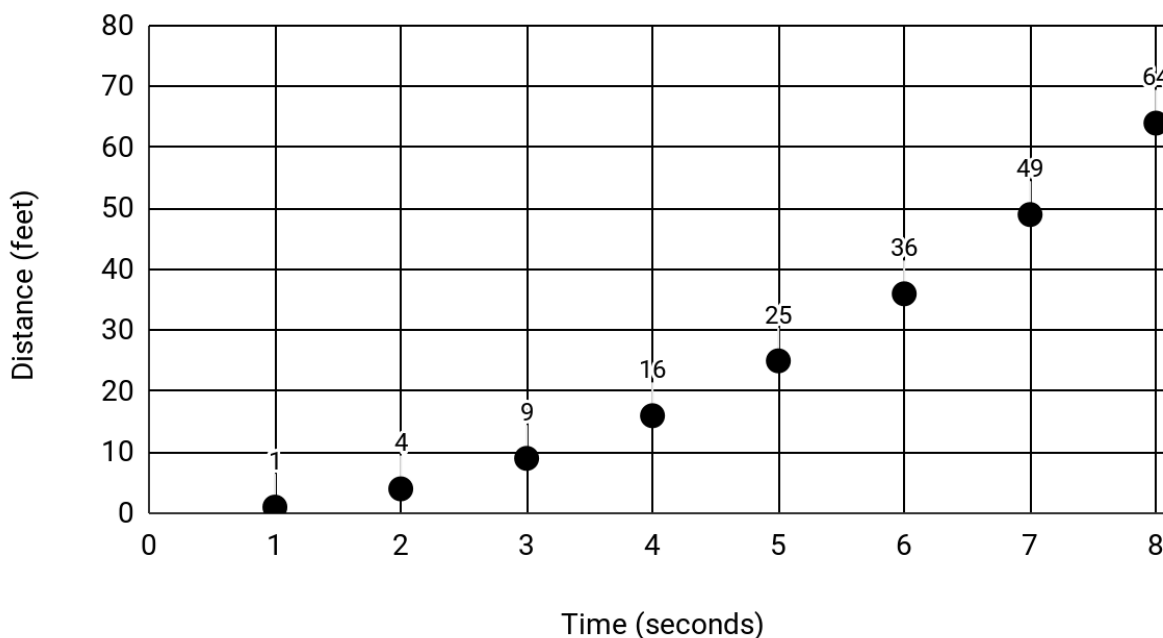
If you look at the change in the outputs in a linear function, the rate of change will be constant (the same number). In the inclined plane experiment, the change in distance (speed) doesn't stay the same. The speed changes as the ball rolls faster and faster down the ramp. The change in speed is called *acceleration*.

- 11) What patterns do you see in how the change in distance (speed) is increasing?

- 12) Is the rate of change constant?
- A. Yes; the inputs are consecutive and the change in the outputs is constant.
  - B. Yes; the inputs are consecutive and the change in the outputs is not constant.
  - C. No; the inputs are consecutive and the change in the outputs is constant.
  - D. No; the inputs are consecutive and the change in the outputs is not constant.

When we look at the rate of change for this table, we can see that the difference between each output is changing as time passes. The change is 1, then 3, then 5, then 7, etc. The rate of change is changing! This tells us that the growth is nonlinear.

Rolling a Ball Down a Ramp



- 13) Connect the circles on the graph with a pencil. Can you make a straight line that goes through all of the circles?

The acceleration (change in speed) of a ball rolling down a ramp is an example of a quadratic function, which is a kind of a nonlinear function. The growing distance is a function of the time spent rolling down the ramp. We will spend more time exploring gravity and other quadratic situations later in the packet.

## Choose Your Salary

I would like to offer you a job for thirty days. You will only need to work one hour each day. The interesting part is that I am going to let you decide how you want to be paid. If you like, I will pay you \$1,000 per day. Your other option is to be paid 1 penny on the first day, 2 pennies on the second day, 4 pennies on the third day, and so on, each day earning twice as many pennies as the day before.



- 14) Which way of being paid would you prefer? Why?
  
  
  
  
  
  
  
  
  
  
- 15) If you choose the thousand-dollar-per-day method, how much money would you make by Day 5? (For example, by Day 4 you would make 4 thousand dollars because  $\$1,000 + \$1,000 + \$1,000 + \$1,000 = \$4,000$ .)
  
  
  
  
  
  
  
  
  
  
- 16) If you choose the doubling penny each day method, how much money would you make by Day 5? (For example, by Day 4 you would make 15 cents because  $\$0.01 + \$0.02 + \$0.04 + \$0.08 = \$0.15$ .)

*Tools of Algebra: Nonlinear Functions (Part 1)*

17) How much will you earn after 9 days with the thousand-dollar-per-day method?

Days	Daily Pay	Calculation	Total Payments
1	\$1,000.00	$1,000.00 + 0$	\$1,000.00
2	\$1,000.00	$1,000.00 + 1,000.00$	\$2,000.00
3	\$1,000.00	$1,000.00 + 2,000.00$	\$3,000.00
4	\$1,000.00	$1,000.00 + 3,000.00$	
5	\$1,000.00		
6	\$1,000.00		
7	\$1,000.00		
8	\$1,000.00		
9	\$1,000.00		

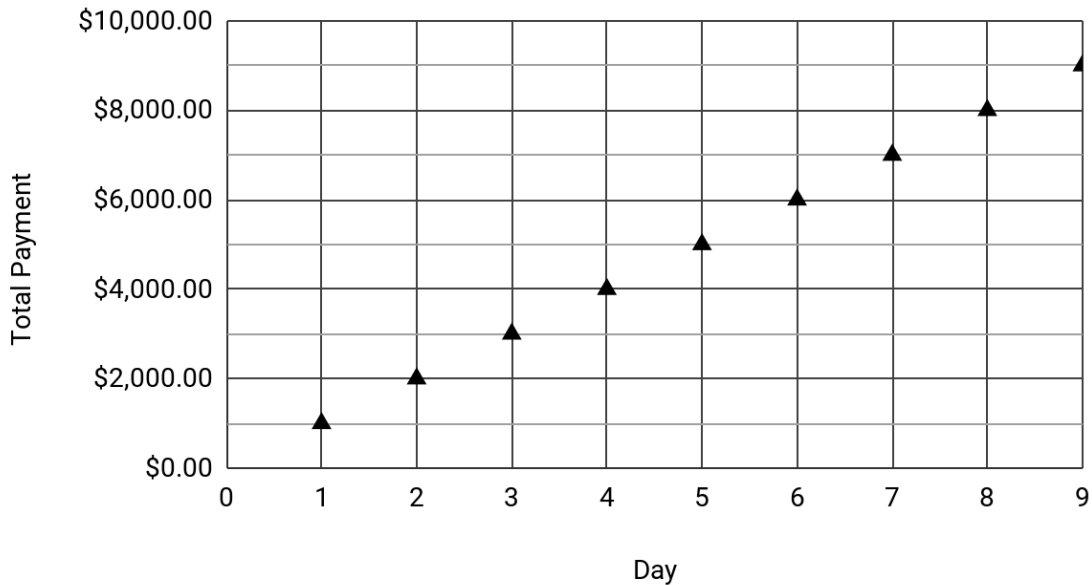
18) How much will you earn after 9 days with the doubling pennies method?

Days	Daily Pay	Calculation	Total Payments
1	\$0.01	-	\$0.01
2	\$0.02	$.01 + .02$	\$0.03
3	\$0.04	$.03 + .04$	\$0.07
4	\$0.08	$.07 + .08$	
5	\$0.16		
6	\$0.32		
7			
8			
9			

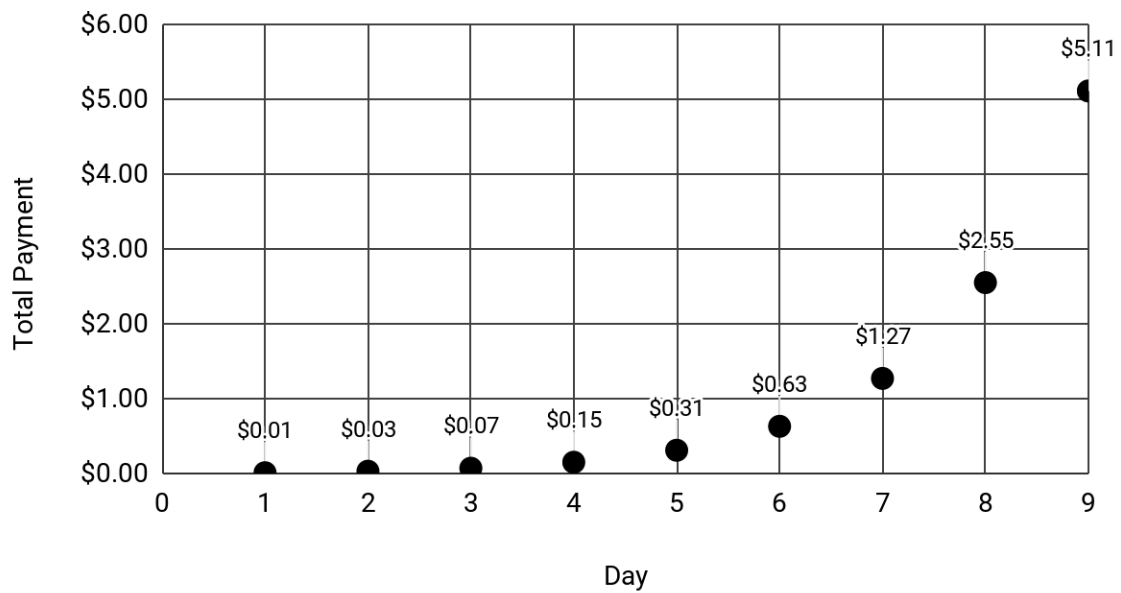
*Tools of Algebra: Nonlinear Functions (Part 1)*

- 19) Below are graphs of each payment option, through Day 9. What do you notice? What predictions can you make about what will happen with the total payments for each option over 30 days?

Thousand Dollars Per Day



Doubling Pennies - Total Payment



*Tools of Algebra: Nonlinear Functions (Part 1)*

- 20) Based on what you know so far, which payment method would you choose? Is it different from the choice you made at the beginning? Include calculations and explain your choice below.

The thousand dollars per day payment option is an example of a linear growth function. The total payments of a thousand dollars a day form a straight line on a graph. The doubling pennies payment option is an example of an exponential growth function, which is a type of nonlinear function. The graph of the total payments of doubling pennies is a curve, not a straight line.

We will return to the doubling pennies and other exponential functions later in the packet.

## Introduction to Nonlinear Functions - Answer Key

- 1) \$6.00
- 2) B and C are nonlinear graphs. The graphs A and B show linear functions.
- 3) The ball is going faster and faster, traveling farther in each unit of time.
- 4) 4 feet
- 5) 9 feet

6)

time (seconds)	distance (feet)
0	0
1	1
2	4
3	9
4	16
5	25
6	36
7	49
8	64

- 7) 100 feet
- 8) We will look at some of the patterns later. For now, we're interested in patterns you see.

Tools of Algebra: Nonlinear Functions (Part 1)

time (seconds)	distance (feet)	change in distance
0	0	
1	1	+1
2	4	+3
3	9	+5
4	16	+7
5	25	+9
6	36	+11
7	49	+13
8	64	+15

9)

10) The ball is not rolling at a constant speed. If it was, the change in distance would be the same between each pair of outputs. Since the change in distance keeps increasing and the amount of time passing is the same, the speed must be increasing.

11) There are different patterns to notice here:

- The changes in distance are the odd numbers: 1, 3, 5, 7, 9, 11, 13, 15, etc.
- There is a difference of 2 between each change in distance.
- The changes in distance are growing by 2 feet each second.

12) D

13) It may seem at first that you can, but the points are not all in a straight line. You would have to keep curving the line in order to connect them all.

14) We want to know what you think.



*Tools of Algebra: Nonlinear Functions (Part 1)*

15) \$5,000

16) \$0.31 (31 cents). This is because  $\$0.01 + \$0.02 + \$0.04 + \$0.08 + \$0.16 = \$0.31$

17)

Day	Daily Pay	Calculation	Total Payments
1	\$1,000.00	$1,000.00 + 0$	\$1,000.00
2	\$1,000.00	$1,000.00 + 1,000.00$	\$2,000.00
3	\$1,000.00	$1,000.00 + 2,000.00$	\$3,000.00
4	\$1,000.00	$1,000.00 + 3,000.00$	\$4,000.00
5	\$1,000.00	$1,000.00 + 4,000.00$	\$5,000.00
6	\$1,000.00	$1,000.00 + 5,000.00$	\$6,000.00
7	\$1,000.00	$1,000.00 + 6,000.00$	\$7,000.00
8	\$1,000.00	$1,000.00 + 7,000.00$	\$8,000.00
9	\$1,000.00	$1,000.00 + 8,000.00$	\$9,000.00

18)

Day	Daily Pay	Calculation	Total Payments
1	\$0.01	-	\$0.01
2	\$0.02	$0.01 + 0.02$	\$0.03
3	\$0.04	$0.03 + 0.04$	\$0.07
4	\$0.08	$0.07 + 0.08$	\$0.15
5	\$0.16	$0.15 + 0.16$	\$0.31
6	\$0.32	$0.31 + 0.32$	\$0.63
7	\$0.64	$0.63 + 0.64$	\$1.27
8	\$1.28	$1.27 + 1.28$	\$2.55
9	\$2.56	$2.55 + 2.56$	\$5.11

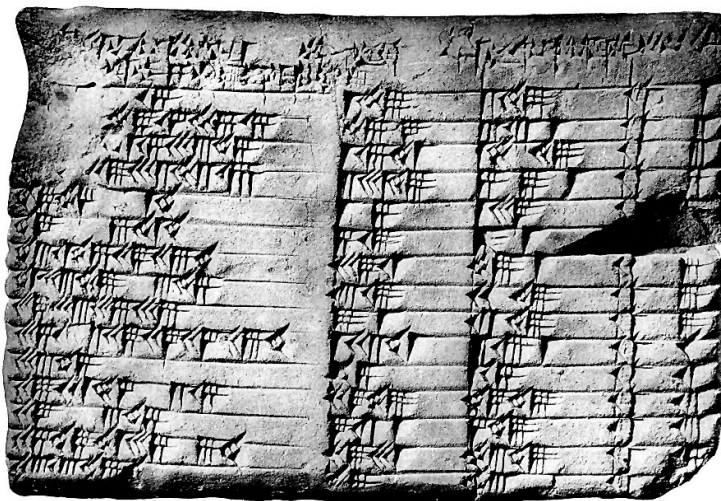
19) The graph of the thousand-dollars-per-day payment method is a straight line. To predict the total payment, we can multiply the number of the day by \$1,000.

The graph of the doubling-pennies method is not a straight line. The amount paid each day doubles each day. The pay is tiny for the first few days, but then keeps getting larger as time goes on. We can predict that the pay would keep doubling as time goes on.

Use the information you have so far to make a decision about which payment method you would choose. Feel free to keep calculating the total pay for each method for more days. We will return to this problem later in the packet.

## Introduction to Quadratic Functions

More than 3000 years ago, the Babylonian people, who lived in present-day Iraq, developed techniques for solving many mathematical problems related to the management of business and agriculture (farming and producing food). They kept track of their calculations on clay tablets. Many of these tablets have been found. After analyzing the tablets, modern mathematicians found that the Babylonians had discovered many tools of algebra. For example, they calculated area and volume, squares and square roots, and understood quadratics and exponential growth (though the idea of a function was something that came much later).



The word *quadratic* comes from the root word *quad*, which means “four.” Geometric figures with four sides are called *quadrilaterals*. The root word *quad* is used in the word *quadratic* because these functions are based on squares, which have four sides.

- 1) On the right is an example of a quadratic function in a table. Find the missing outputs.

The rule for this function table ( $Out = In \times In$ ) could also be written as  $y = x \cdot x$  since  $x$  can represent the input to a function and  $y$  can represent the output. The function could also be written as  $y = x^2$  or  $f(x) = x^2$ . Each of these equations are equivalent to each other.

- 2) If  $y = x^2$  and  $x$  is 7, what value of  $y$  makes the equation true?

Rule: $Out = In \times In$	
In	Out
1	1
2	4
3	9
4	
5	

## Number Patterns

In the high school equivalency test, you will need to figure out whether you are looking at a linear, quadratic, or exponential function. You may be given a number pattern, an equation, an input/output table, a graph, or a visual pattern. In this packet we will practice analyzing all of these so that you will understand what kind of function you are looking at.

We will start by looking at how numbers grow or *decay* (get smaller) in number patterns. In this exercise, find a pattern in the numbers, write in the next two numbers in the pattern, and then describe the pattern.

Continue the pattern.

Describe the pattern.

3) 18, 16, 14, 12, 10, 8, ...

Subtract 2 each time.

4) 7, 14, 21, 28, \_\_\_\_\_, \_\_\_\_\_, ...

5) 1, 4, 9, 16, \_\_\_\_\_, \_\_\_\_\_, ...

6) 1, 2, 4, 8, \_\_\_\_\_, \_\_\_\_\_, ...

7) 1, 3, 6, 10, \_\_\_\_\_, \_\_\_\_\_, ...

8) 1, 3, 9, 27, \_\_\_\_\_, \_\_\_\_\_, ...

9) 3, 9, 19, 33, \_\_\_\_\_, \_\_\_\_\_, ...

10) 3, 7, 13, 21, \_\_\_\_\_, \_\_\_\_\_, ...

11) 3, 6, 9, 12, \_\_\_\_\_, \_\_\_\_\_, ...

12) 2, 6, 12, 20, \_\_\_\_\_, \_\_\_\_\_, ...

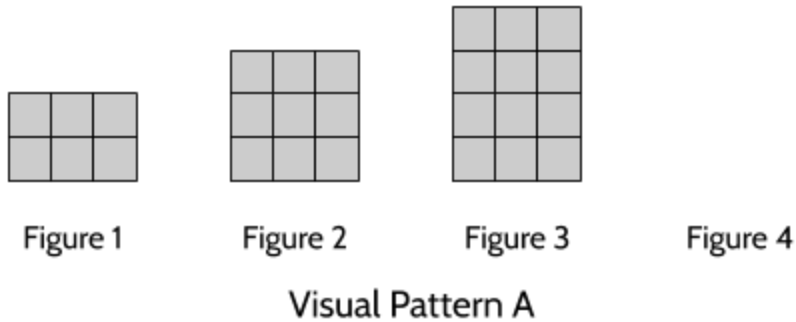
13) 16, 8, 4, 2, \_\_\_\_\_, \_\_\_\_\_, ...

## Visual Patterns

Visual patterns are another way to see how different kinds of growth happen. For each of the following visual patterns, count the number of pieces and see if you can figure out the growth pattern.

14) Look at Visual Pattern A, then follow the instructions below.

a) Draw Figure 4.



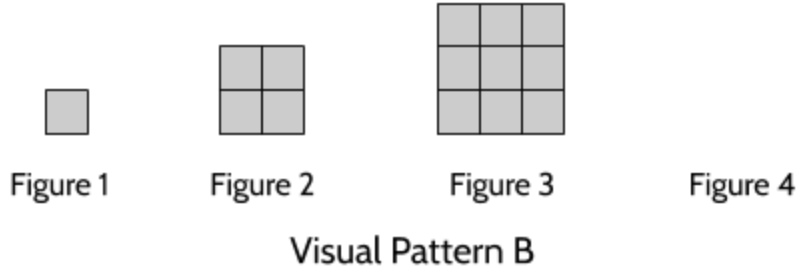
b) Complete the table.

Figure	Number of Squares
1	6
2	9
3	
4	
5	
6	

c) How is this visual pattern growing?

15) Look at Visual Pattern B, then follow the instructions below.

a) Draw Figure 4.

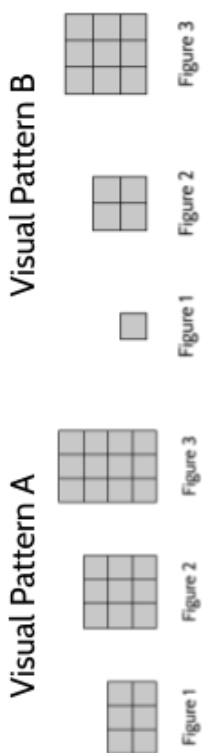
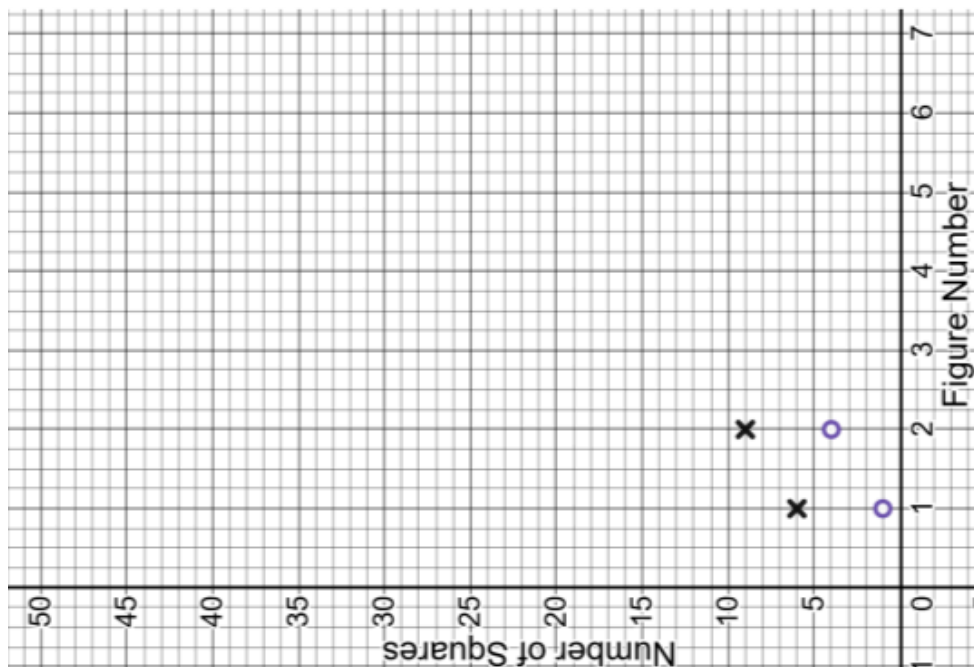


b) Complete the table.

Figure	Number of Squares
1	
2	
3	
4	
5	
6	

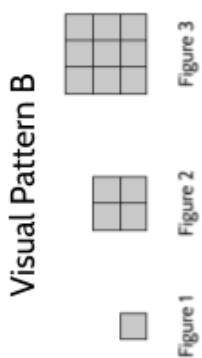
c) How is this visual pattern growing?

16) Turn the sheet and follow the instructions.



Use **X**'s to plot the data from the table below.

Figure	Number of Squares
1	6
2	9
3	12
4	15
5	
6	
7	



Use **O**'s to plot the data from the table below.

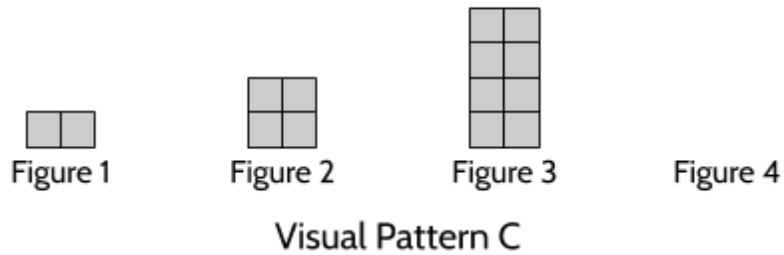
Figure	Number of Squares
1	1
2	4
3	9
4	16
5	
6	
7	

Complete the tables and plot the X's and O's.

What do you notice when you look at the tables and graphs of these two visual patterns?

17) Look at Visual Pattern C, then follow the instructions below.

a) Draw Figure 4.



b) Complete the table.

Figure	Number of Squares
1	
2	
3	
4	
5	
6	

c) How is this visual pattern growing?

*Tools of Algebra: Nonlinear Functions (Part 1)*

Below are input/output tables for each of the three visual patterns above.

**A**

Figure	Number of Squares
1	6
2	9
3	12
4	15
5	18
6	21

**B**

Figure	Number of Squares
1	1
2	4
3	9
4	16
5	25
6	36

**C**

Figure	Number of Squares
1	2
2	4
3	8
4	16
5	32
6	64

18) What do you notice when you look at the input/output tables side by side?

19) Which of these tables represents a linear function? Why?

20) Which tables represent a nonlinear function? Why?

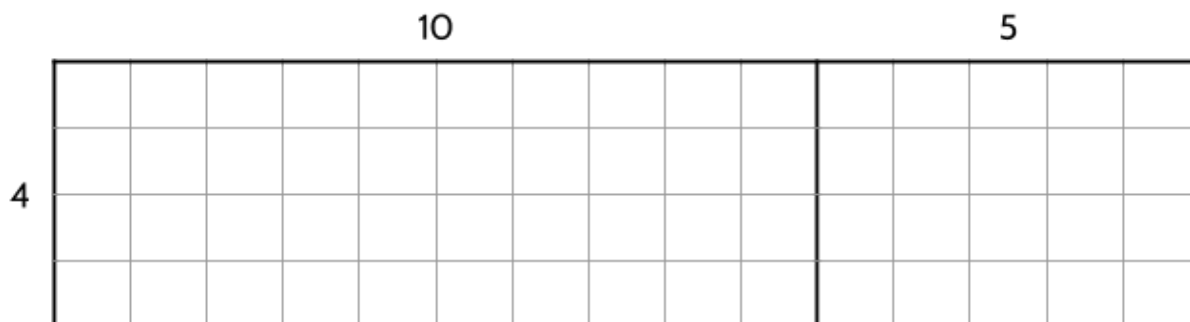
One of these functions is linear, one is quadratic, and one is exponential. We will examine each of these functions later in the packet.



## Area Models and Quadratic Functions

An area model can be used to understand multiplication. We will use it here to see how multiplication is an important part of how quadratic functions work.

In **Tools of Algebra: Expressions, Equations, and Inequalities**, we used area models to write equations for the area of rectangles. Here is an example of an area model of multiplication:



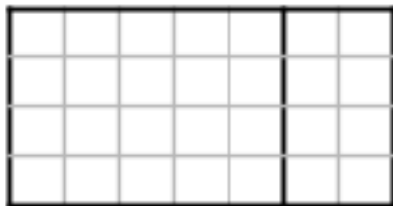
$$(4 \cdot 10) + (4 \cdot 5) = ?$$

$$40 + 20 = 60$$

The area of a rectangle (or the number of small squares that fill the space) can be found by multiplying the length by the width. Using the distributive property<sup>2</sup>, we can find the area of the rectangle on the left by multiplying 4 by 10. Then we can find the area of the rectangle on the right by multiplying 4 by 5. Finally, we can find the total area by adding those two products ( $40 + 20$ ) to get 60. This is one way of multiplying  $4 \times 15$  to get 60.

21) Which expression could be used to calculate the area of this rectangle?

- A.  $4(5 + 2)$
- B.  $4(3 + 5)$
- C.  $2(4 + 5)$
- D.  $5(4 + 2)$



<sup>2</sup> You can find practice on this topic in **Tools of Algebra: Expressions, Equations, and Inequalities**.

The expression  $4(5 + 2)$  gives the area of the rectangle above. In the area model examples so far, we knew the length of the sides of the rectangle and used the side lengths to find the area. What if we knew the area of the rectangle, but didn't know the length of one of the sides?

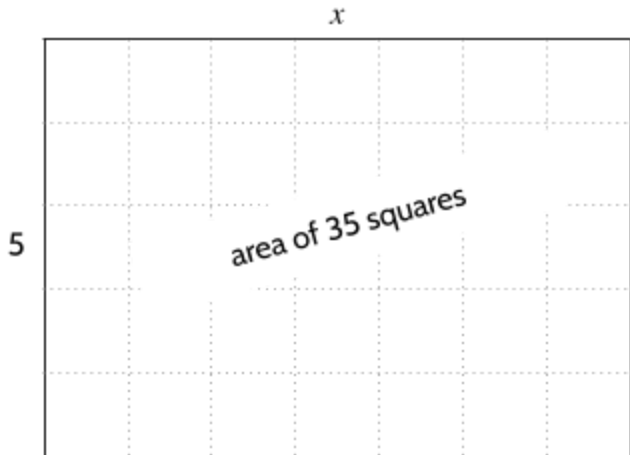
- 22) What is the length of the missing side of this rectangle?

Equation:

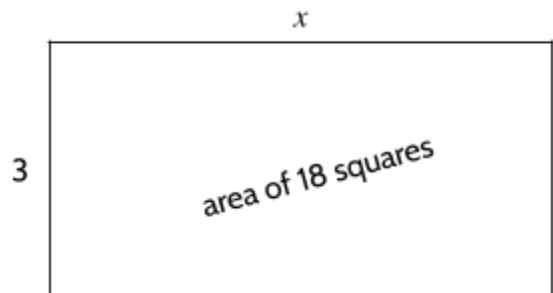
$$5 \cdot x = 35$$

$$x = \underline{\hspace{2cm}}$$

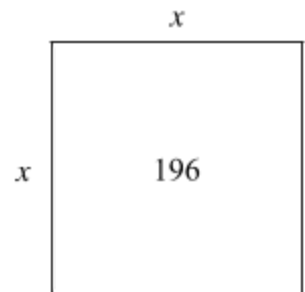
The area model on the right could be represented by the equation  $5 \cdot x = 35$ , meaning some number multiplied by 5 is equal to 35.



- 23) What is the value of  $x$  in this rectangle?

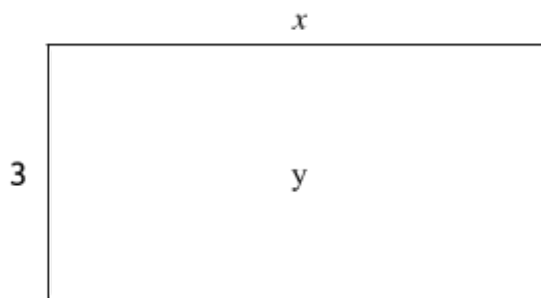


- 24) What value of  $x$  would result in an area of 196 square units for this rectangle<sup>3</sup>?



<sup>3</sup> Hint: If a variable is used twice in a drawing or equation, it represents the same number. If the length and width of the rectangle are  $x$ , they must be the same length. Since the length and width are the same, this figure is a square.

We could also imagine a situation where we don't know the length of one side of the rectangle or the area of the rectangle. The width of the rectangle on the right is 3, but we don't know the length,  $x$ , or the area,  $y$ .



25) Which equation below correctly represents the area of the rectangle?

A.  $3 + x = y$

B.  $3y = x$

C.  $x^2 + 3 = y$

D.  $3x = y$

26) If  $x$  is 1.5, what is the value of  $y$ ?

27) If  $y$  is 33, what is the value of  $x$ ?

The rectangle above could represent a function. For each missing side length,  $x$ , there is a corresponding area,  $y$ . The area of the rectangle is a function of the side length of 3 and the missing side length,  $x$ . For example, if the missing side length is 2, then the area is 6.

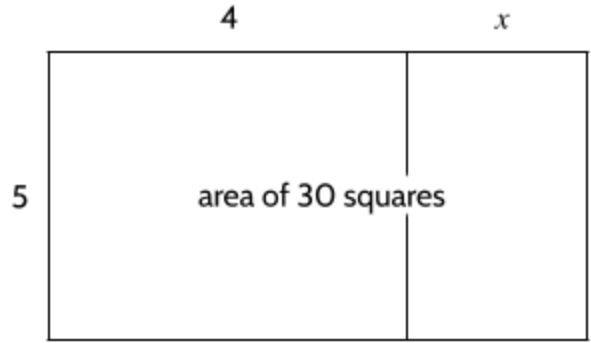
A function table could be used to show the missing length and the area for many rectangles with a width of 3 units.

28) Complete the table for missing inputs and outputs.

Rule: $y = 3x$	
$x$	$y$
11	
2	6
5	
$\frac{1}{3}$	
	.75

*Tools of Algebra: Nonlinear Functions (Part 1)*

In other situations, we might know part of the length of a rectangle, but are missing some information. In the rectangle on the right, we know the width of the rectangle, but there is a missing part of the length,  $x$ .

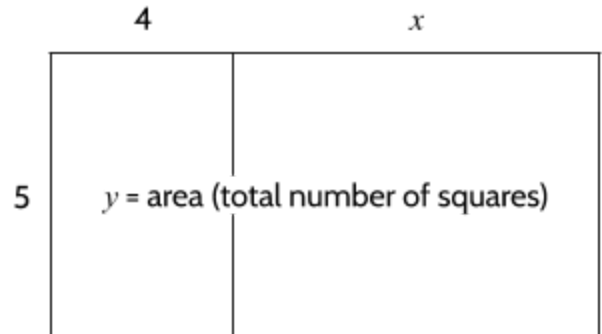


29) Which equation below does not correctly represent the area of the rectangle?

- A.  $5(4 + x) = 30$
- B.  $(5 \cdot 4) + (5 \cdot x) = 30$
- C.  $20 + 5x = 30$
- D.  $4(5 + x) = 30$

30) What is the missing length,  $x$ ? Explain how you know.

In this rectangle, there is a missing part of the length,  $x$ , and we don't know the area,  $y$ , either.

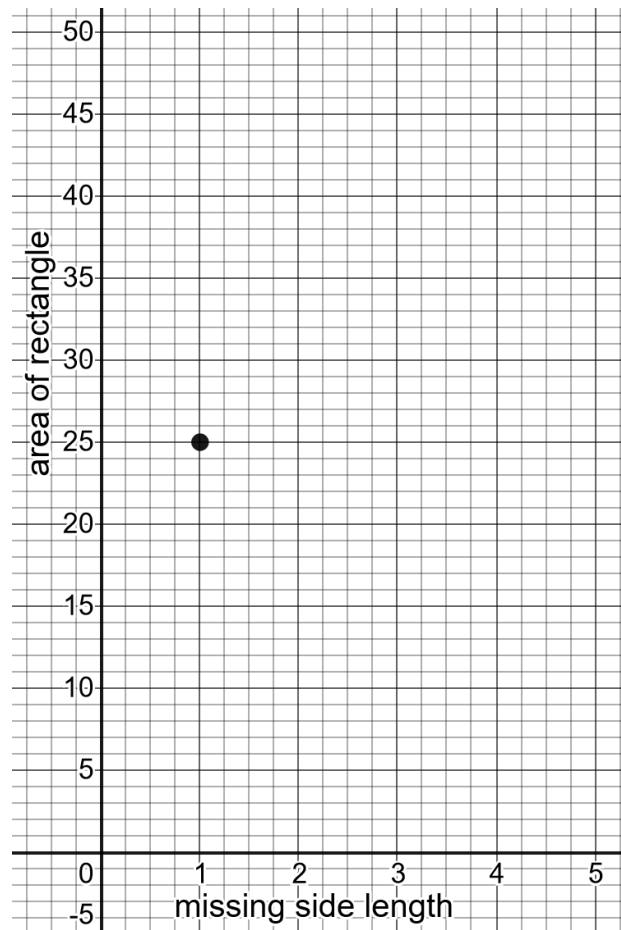
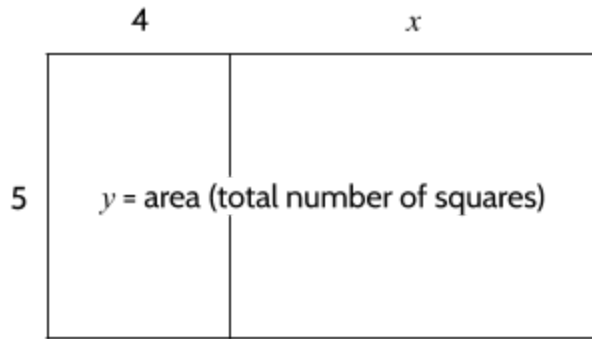


31) Write an equation that represents the area of the rectangle.

Tools of Algebra: Nonlinear Functions (Part 1)

- 32) Write your equation/rule in at the top of the function table below, then complete the table. Finally, graph the points of the table on the graph.

Rule:	
$x$	$y$
0	
1	25
2	
3	
4	
5	



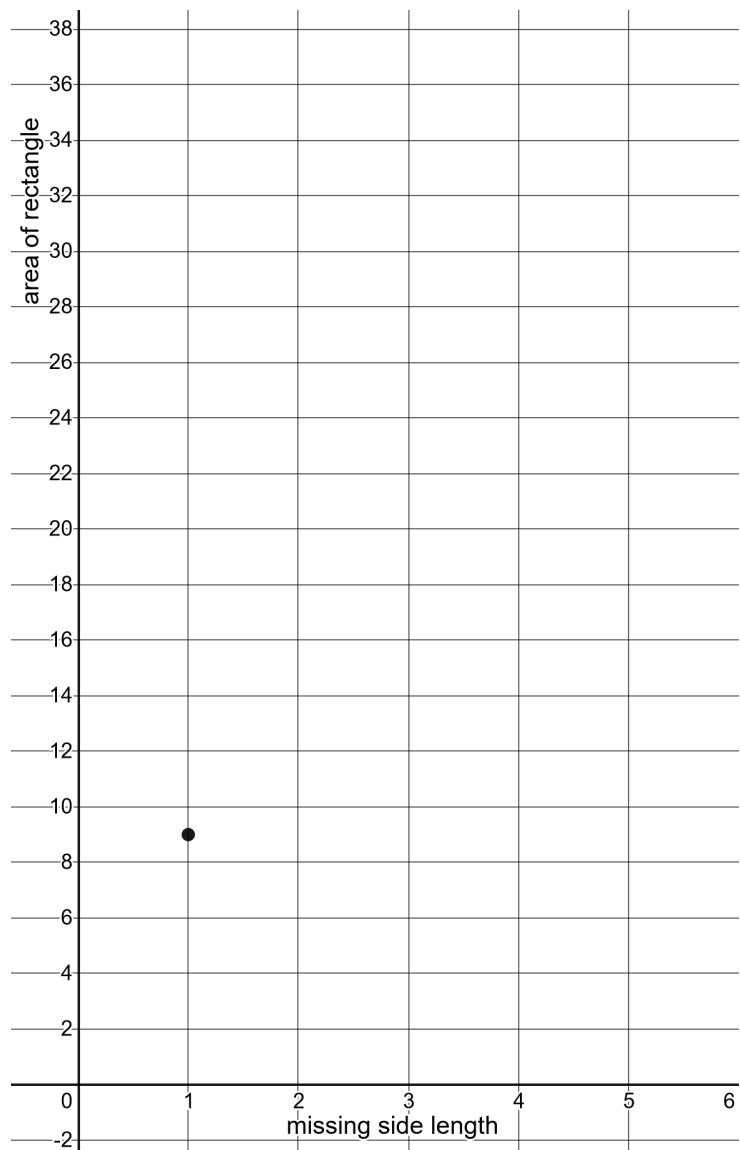
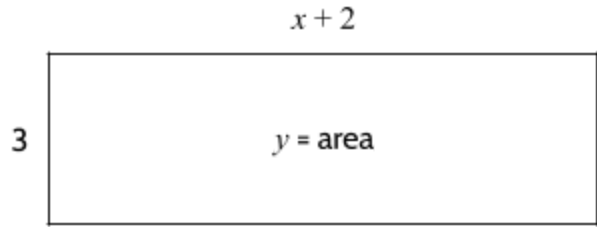
- 33) Is this a linear or a nonlinear function? Why?

Tools of Algebra: Nonlinear Functions (Part 1)

34) Look at the area model below, then complete the following:

- Write an equation/rule in at the top of the function table
- Complete the table
- Graph the points of the table on the graph

Rule:	
$x$	$y$
0	
1	
2	
3	
4	
5	

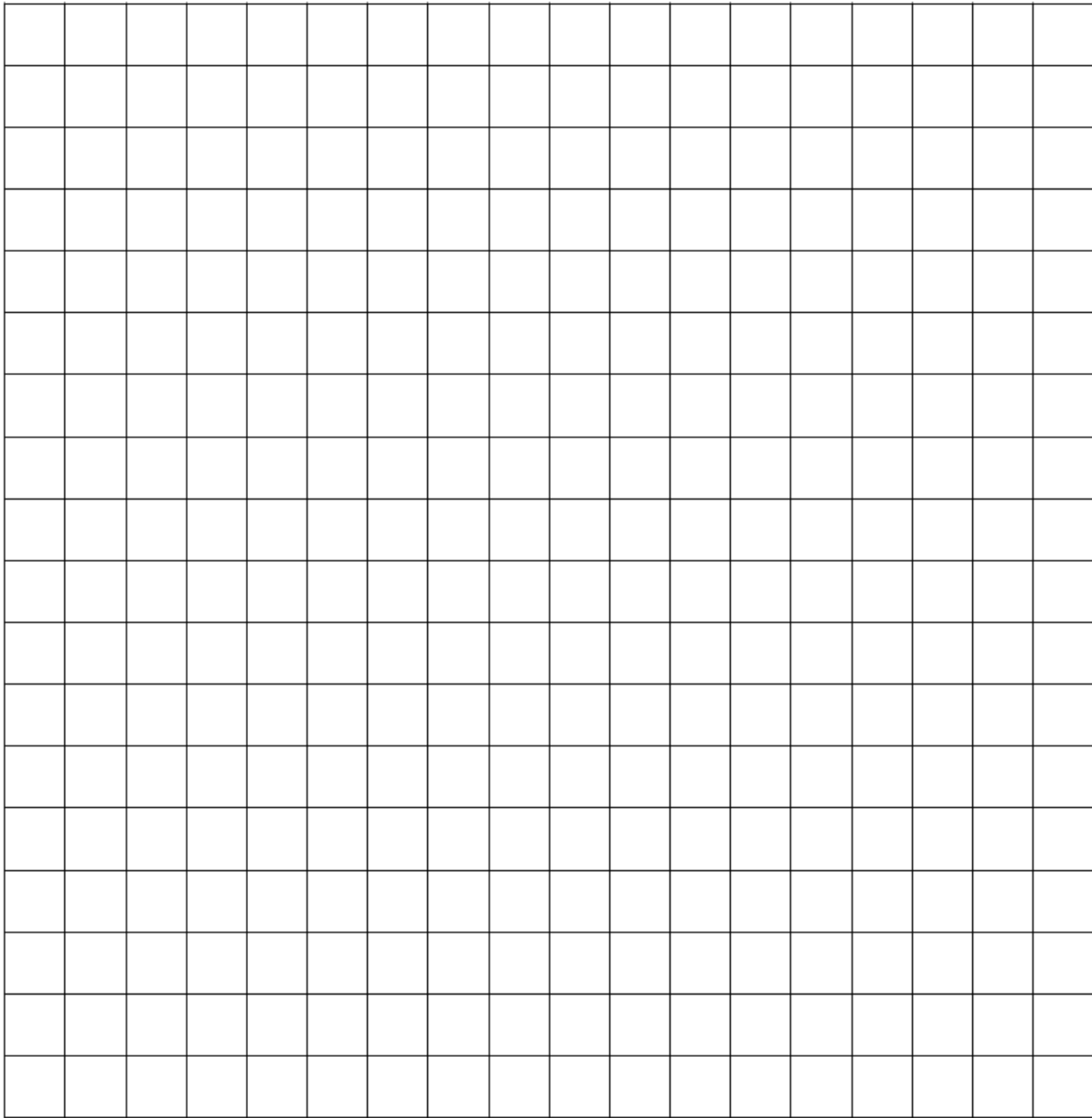


*Tools of Algebra: Nonlinear Functions (Part 1)*

- 35) We often see math questions that include a combination of geometry and algebra. For example, you might be asked the following question:

A rectangle has an area of 60 square inches and a length that is 4 inches longer than its width. What are the dimensions<sup>4</sup> of the rectangle?

Use the graph paper below to find the rectangle.



---

<sup>4</sup> *Dimensions* means the measurements of a geometric figure. In this case, it means the length and width of the rectangle.

*Tools of Algebra: Nonlinear Functions (Part 1)*

We could also use a table to find a solution to the problem. We will use the table to record our attempts, but we have to remember that the length has to be 4 inches longer than the width. When we find a rectangle which has a length 4 inches longer than the width and also has an area of 60 square inches, we will know that we have the right answer.

Let's try different measurements to see what works. Maybe the width is 3 inches and the length is 7 inches? That would give us an area of 21 square inches. Too small.

How about a width of 10 inches and a length of 14 inches? The area would be 140 square inches. Too big.

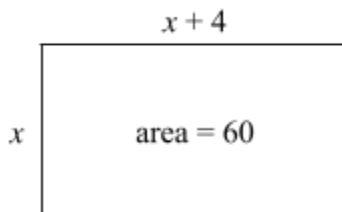
- 36) Use different lengths and widths until you get an area of 60 square inches. A few measurements are suggested to get you started.

Width	Length	Calculation	Area
3 in.	7 in.	$3 \cdot 7$	21 sq. in.
10 in.	14 in.	$10 \cdot 14$	140 sq. in.
4 in.	8 in.		
9 in.			



## Tools of Algebra: Nonlinear Functions (Part 1)

We can also use variables to understand this problem. Here is a sketch of the rectangle using the variable  $x$  to represent the width:



Width	Length	Calculation	Area
$x$	$x + 4$	$x(x + 4)$	60

The length is always 4 inches more than the width, so the expression  $x + 4$  represents the length. The width ( $x$ ) times the length ( $x + 4$ ) is equal to the area (60). The area of this rectangle can be shown by the equation below:

$$x(x + 4) = 60$$

Using the distributive property, we can multiply  $x$  by  $x + 4$  to get  $x^2 + 4x$ , which means the equation could also be written as:

$$x^2 + 4x = 60$$

Here is the original problem, which we have represented with the equation above:

A rectangle has an area of 60 square inches and a length that is 4 inches longer than its width. What are the dimensions of the rectangle?

One way to solve the problem is to write the equation  $x(x + 4) = 60$  and then find a value for  $x$  that makes the equation true. The value of  $x$  that makes the equation true is 6, since  $6(6 + 4) = 60$ , so the answer to the question is: A width of 6 inches and a length of 10 inches.

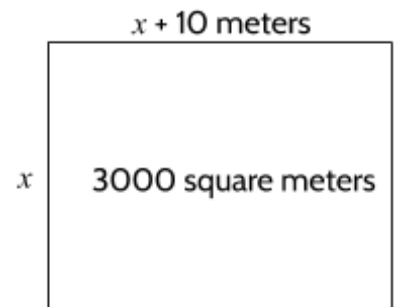
37) Does  $x = 6$  also make the equation  $x^2 + 4x = 60$  true? Explain below.

*Tools of Algebra: Nonlinear Functions (Part 1)*

- 38) A rectangle is 3 inches longer than it is wide. The area of the rectangle is 54 square inches. What are the dimensions of the rectangle?
- 39) A rectangle is twice as long as it is wide. The area of the rectangle is 128 square inches. Which equation correctly represents the area of the rectangle?
- A.  $2w = 128$
  - B.  $w \cdot 2w = 128$
  - C.  $w + 2w = 128$
  - D.  $2w - w = 128$
- 40) Use any method to find the length and width of the rectangle described above.

The problem below is a type of quadratic problem that ancient Babylonian and Islamic mathematicians were able to solve (though they didn't use  $x$  as a variable or *meters* as a measurement).

- 41) A rectangular piece of land has a length that is 10 meters longer than its width and a total area of 3000 square meters. Can you find the value of  $x$  that makes this relationship true?

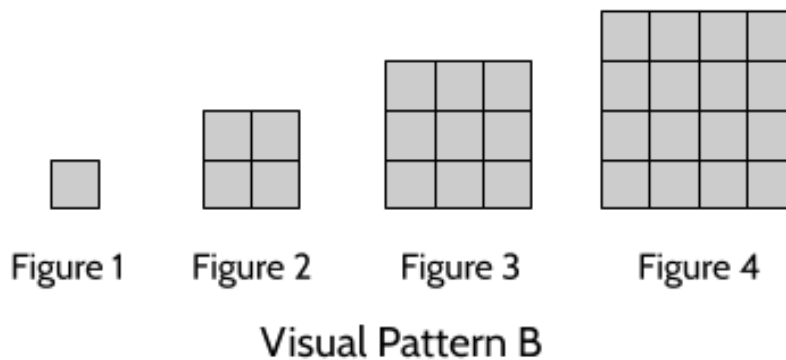


### Where's the Square?

The word quadratic comes from the root word quad, which means “four.” Geometric figures with four sides are called quadrilaterals. The root word *quad* is used in the word *quadratic* because these functions are based on squares, which have four sides.

Take a look at this visual pattern from the previous section.

The number of squares in each figure (1, 4, 9, 16, etc.) can be predicted with the function rule  $y = x^2$  where  $x$  is the figure number and  $y$  is the number of squares in the figure.

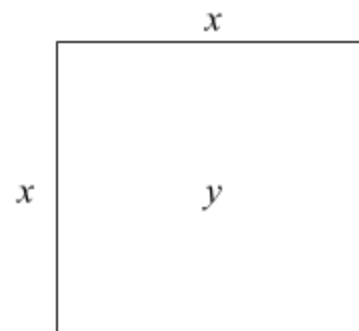


- 42) The quadratic function  $y = x^2$  predicts that there will be 25 squares in Figure 5 of this pattern. Draw a picture to see if this is true.

Rule:	
$y = x^2$	
$x$	$y$
1	1
2	4
3	9
4	16
5	25

You can think of  $y = x^2$  as the area formula for squares. The area of a square is a function of the length of its sides. If  $x$  is the length of each side of a square, then  $y$  is the area of the square.

- 43) If  $x$  is 15, what is  $y$ ?
- 44) If  $x$  is 7.5, what is  $y$ ?
- 45) If  $y$  is 169, what is  $x$ ?



## Tools of Algebra: Nonlinear Functions (Part 1)

$x(x + 4)$  and  $x^2 + 4x$  are examples of quadratic expressions. They are named *quadratic* because the  $x \cdot x$  and  $x^2$  terms are related to squares. But where are the squares in these expressions?

The rectangles on the right show  $x(x + 4)$  and  $x^2 + 4x$  (the same figures since the expressions are equivalent). You can see squares if you look at rectangles drawn for different values of  $x$ . The  $x^2$  section of each rectangle is shaded.

Here are some other examples of quadratic expressions:

$$x^2 \qquad 3x^2 - 15x + 6$$

$$x^2 + 12 \qquad (x + 2)(x - 2)$$

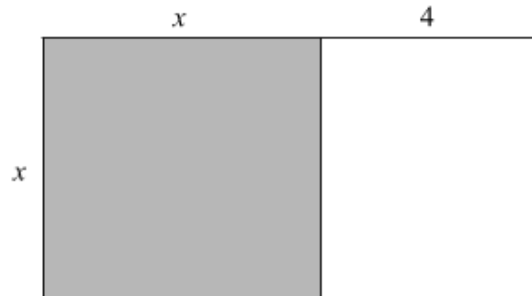
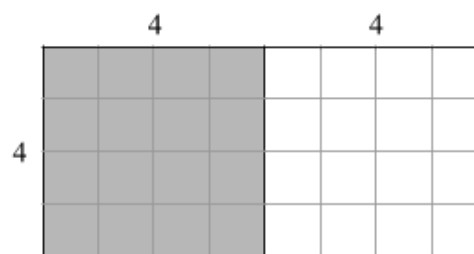
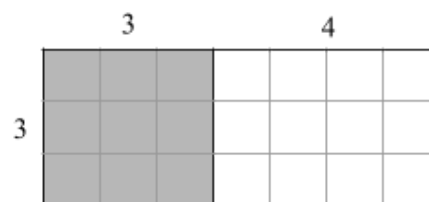
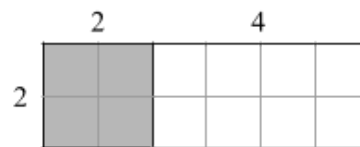
$$x^2 + 6x + 9 \qquad x(x - 2)$$

These expressions are not quadratic:

$$x + 5 \qquad 2^x$$

$$3x \qquad x - 7$$

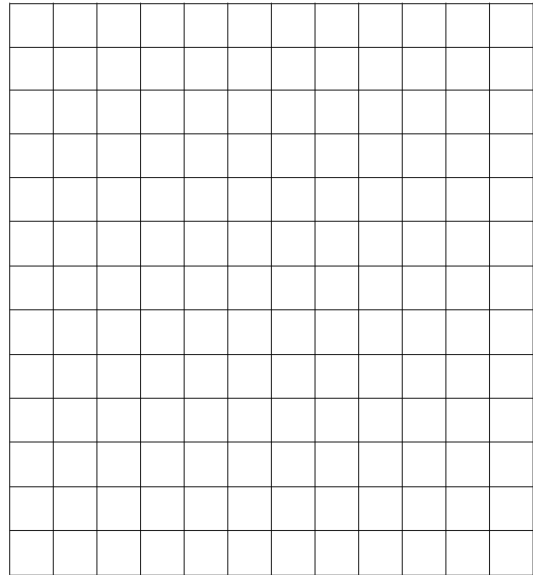
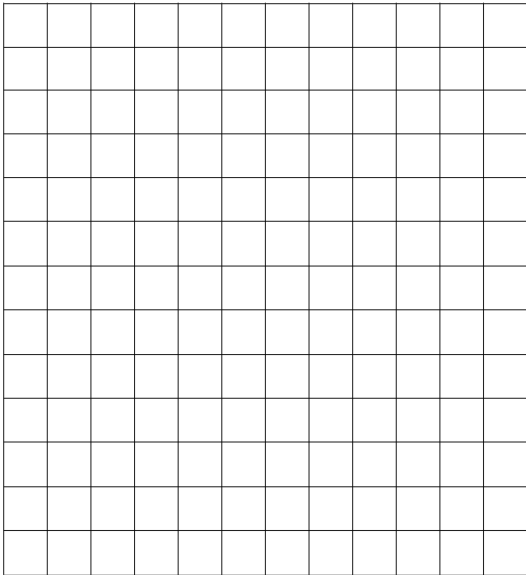
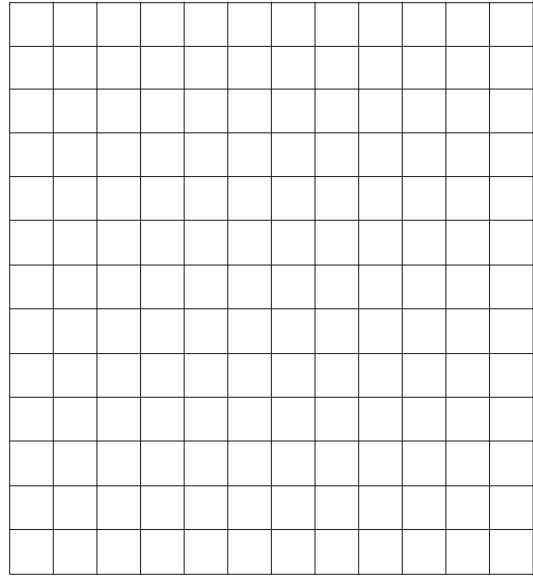
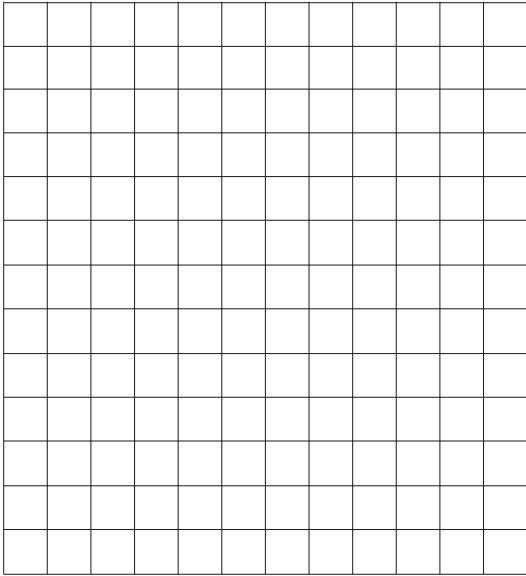
$$x + 3x \qquad x^3 + x^2$$



- 46) Based on these examples, how would you define a quadratic function?

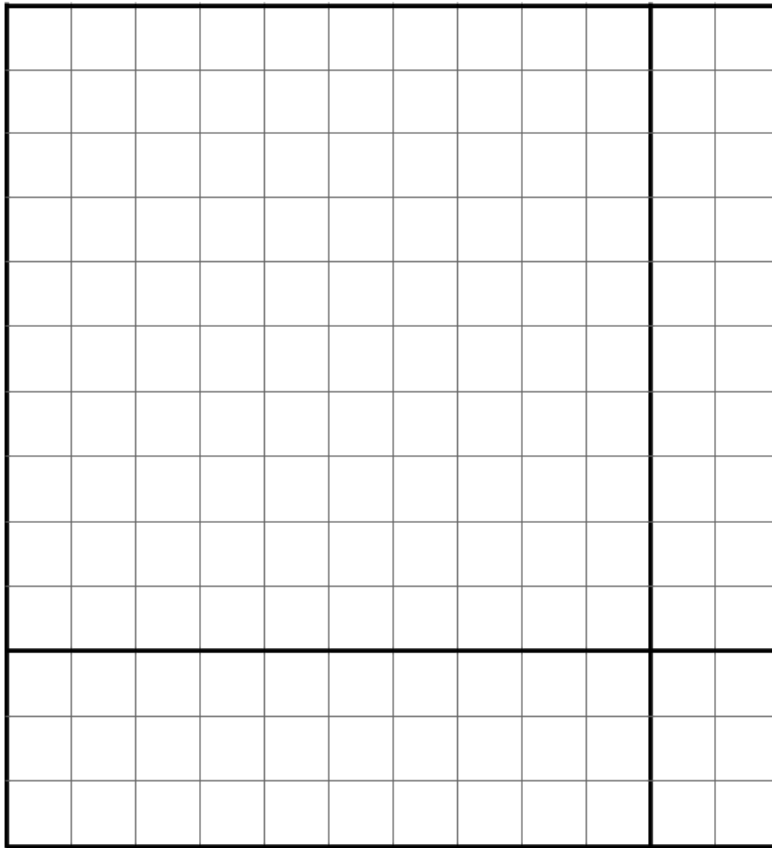
*Tools of Algebra: Nonlinear Functions (Part 1)*

- 47) How many different ways can you find the area of this rectangle? Be creative. (The rectangle is repeated four times so that you can try different methods.)



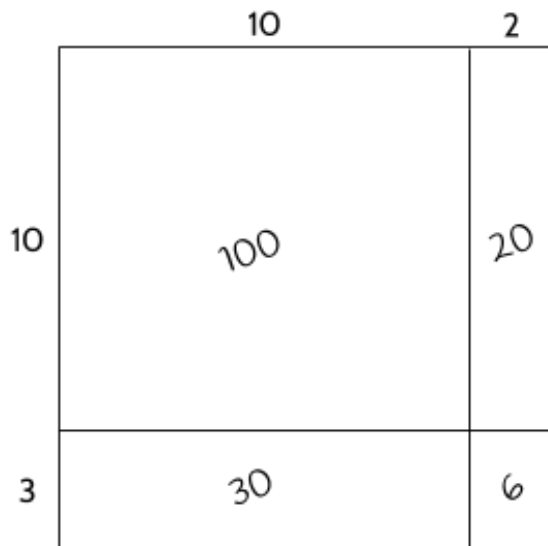
*Tools of Algebra: Nonlinear Functions (Part 1)*

- 48) What if the rectangle were divided in this way? How could you find out the area without counting each square?



*Tools of Algebra: Nonlinear Functions (Part 1)*

Below is an abstract area model that shows the area of each of the smaller rectangles. Adding the areas of the four small rectangles together gives you the area of the large rectangle.

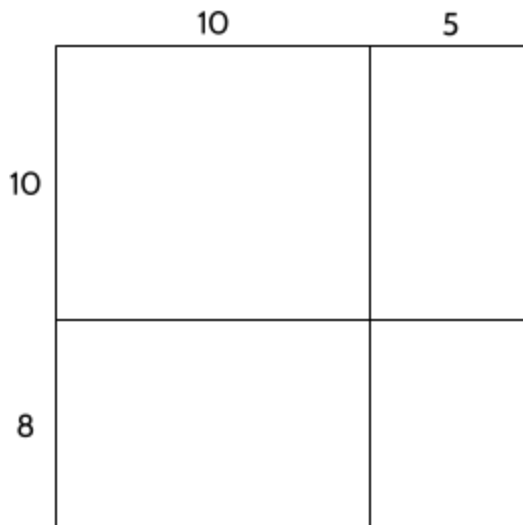


Adding up the area of the small rectangles, starting on the bottom right:

$$\begin{array}{r} 2 \cdot 3 \longrightarrow 6 \\ 2 \cdot 10 \longrightarrow 20 \\ 10 \cdot 3 \longrightarrow 30 \\ 10 \cdot 10 \longrightarrow +100 \\ \hline 156 \end{array}$$

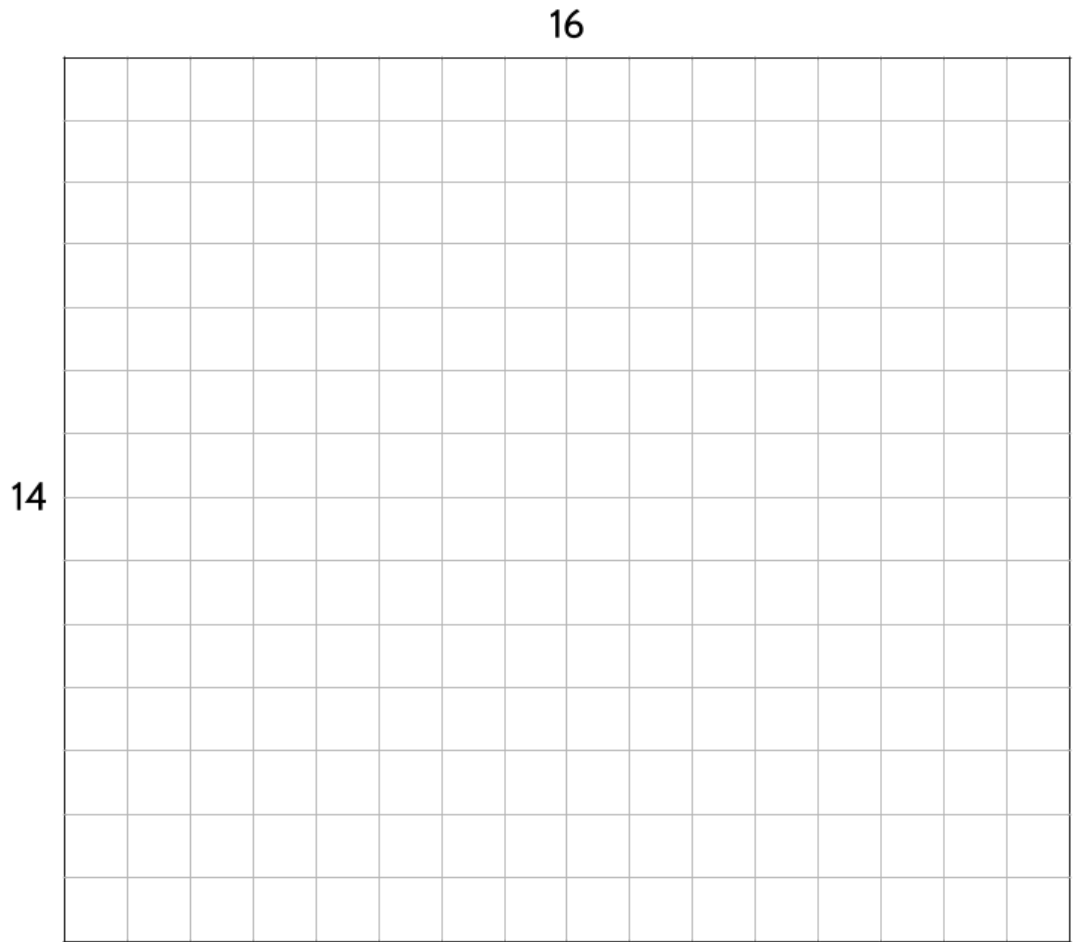
The dimensions of the rectangle above are 13 by 12, which gives a total area of 156 squares. 156 is also the product of  $13 \times 12$ . Try the next rectangle on your own.

- 49) Write in the areas of the different pieces of the rectangle below, then calculate the area of the whole figure.



*Tools of Algebra: Nonlinear Functions (Part 1)*

- 50) Draw lines to break this large rectangle into smaller rectangles, then find the area of the whole rectangle.



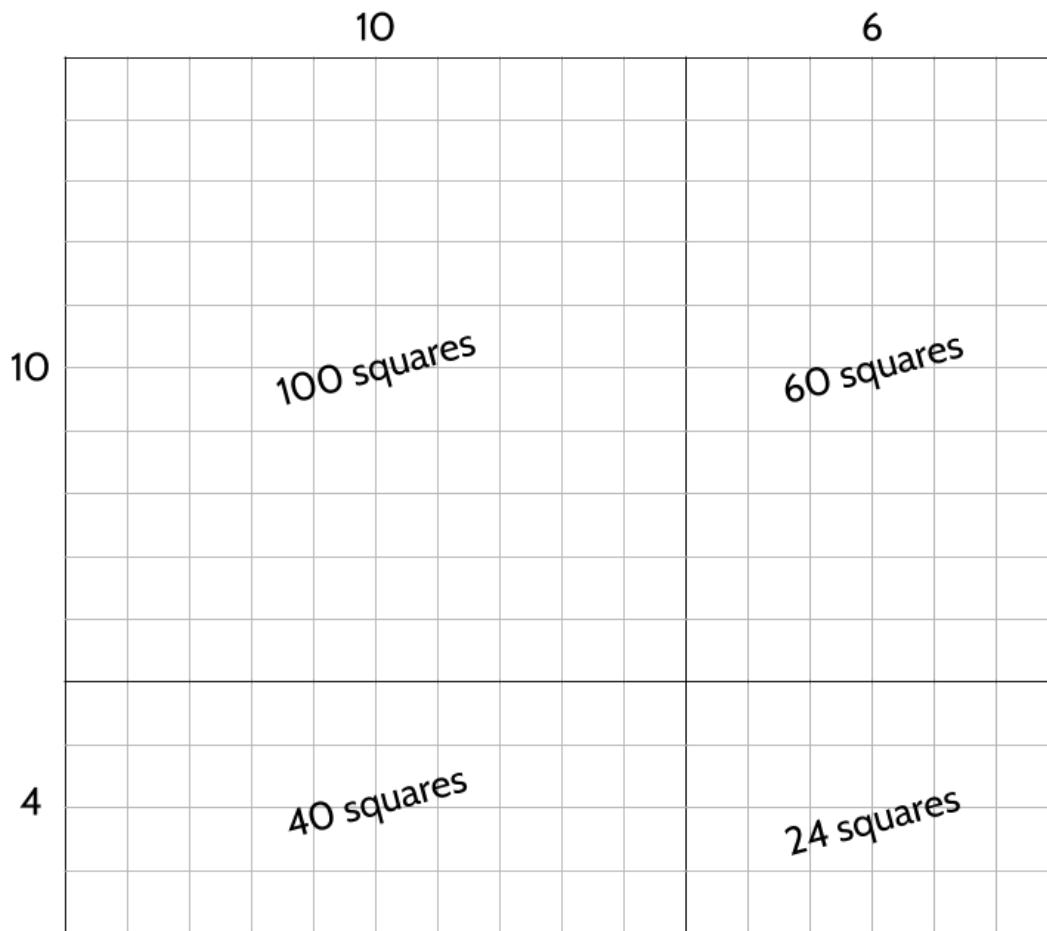
How many small rectangles did you break the large rectangle into?

Show your calculations here:



### Tools of Algebra: Nonlinear Functions (Part 1)

There are many ways to break up the  $14 \times 16$  rectangle into smaller rectangles, in order to calculate the total area. We split the height into  $10 + 4$  and the width into  $10 + 6$ .



The areas of the smaller rectangles can be calculated by multiplying the length and width of the smaller rectangles.

#### **$14 \times 16$ using rectangles**

$$\begin{aligned} & (10 + 4)(10 + 6) = \\ & (10 \cdot 10) + (10 \cdot 6) + (4 \cdot 10) + (4 \cdot 6) = \\ & (100 + 60 + 40 + 24) = 224 \end{aligned}$$

## Tools of Algebra: Nonlinear Functions (Part 1)

Breaking up the area of a rectangle into pieces is similar to a method of multiplying called *partial product multiplication*, which is how many young people are currently learning how to do multiplication. Partial product multiplication lets you calculate the *products* (result of multiplication) one piece at a time. That is why they are called *partial*, since you calculate them one part at a time. One benefit to partial product multiplication is that you don't have to remember to "carry" numbers when you multiply. It can also be easier to organize multiplication with large numbers, since you don't have to remember as many numbers.

Here's an example with  $18 \times 15$ .

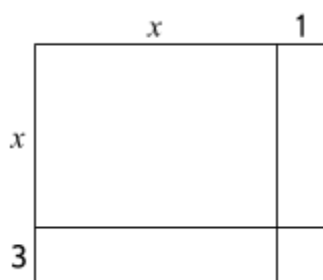
Partial Product Multiplication	Standard Multiplication
$\begin{array}{r} 18 \\ \times 15 \\ \hline \end{array}$	$\begin{array}{r} 18 \\ \times 15 \\ \hline \end{array}$
<div style="display: flex; align-items: center;"> <div style="margin-right: 10px;">partial products</div> <div style="font-size: 3em;">{</div> <div style="margin-left: 10px;"> <math display="block">\begin{array}{r} 40 \quad (5 \times 8) \\ 50 \quad (5 \times 10) \\ 80 \quad (10 \times 8) \\ 100 \quad (10 \times 10) \\ \hline 270 \end{array}</math> </div> </div>	$\begin{array}{r} 90 \\ 180 \\ \hline 270 \end{array}$

Try completing these calculations using partial product multiplication.

<p>51)</p> $\begin{array}{r} 23 \\ \times 17 \\ \hline \end{array}$ <div style="margin-left: 100px;"> <math>(7 \times 3)</math>  <math>(7 \times 20)</math>  <math>(10 \times 3)</math>  <math>(10 \times 20)</math> </div> <hr style="border-top: 1px dotted black;"/>	<p>52)</p> $\begin{array}{r} 16 \\ \times 35 \\ \hline \end{array}$ <div style="margin-left: 100px;"> <math>(5 \times 6)</math>  <math>(5 \times 10)</math>  <math>( \quad \times \quad )</math>  <math>( \quad \times \quad )</math> </div> <hr style="border-top: 1px dotted black;"/>
<p>53)</p> $\begin{array}{r} 14 \\ \times 24 \\ \hline \end{array}$ <div style="margin-left: 100px;"> <math>(4 \times 4)</math>  <math>( \quad \times \quad )</math>  <math>( \quad \times \quad )</math>  <math>( \quad \times \quad )</math> </div>	<p>54)</p> $\begin{array}{r} 25 \\ \times 13 \\ \hline \end{array}$

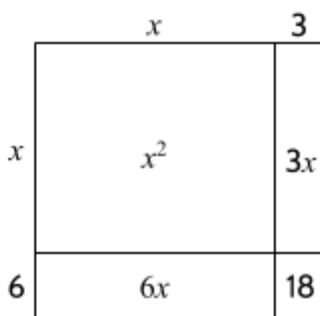
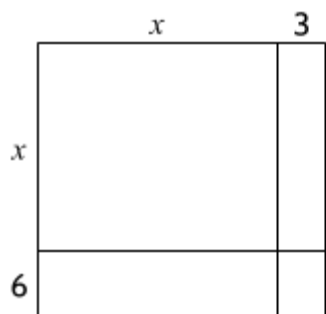
Tools of Algebra: Nonlinear Functions (Part 1)

In the rectangle below, we substituted  $x$  as a variable part of the width and length. This means that the value for  $x$  might be any number. We don't know the area of the rectangle, but we can use multiplication to find an algebraic expression for the area.



55) Using  $(x + 3)$  as the width and  $(x + 1)$  as the length, what is the area of this rectangle?

Using a different rectangle as an example below, we show how you can express the area of the rectangle with variables and algebra. Review the steps below, then look back at the previous question.



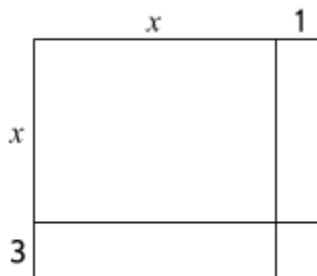
Adding up the area of the small rectangles, starting on the bottom right:

$$\begin{array}{r}
 3 \cdot 6 \longrightarrow 18 \\
 3 \cdot x \longrightarrow 3x \\
 x \cdot 6 \longrightarrow 6x \\
 x \cdot x \longrightarrow +x^2 \\
 \hline
 x^2 + 6x + 3x + 18
 \end{array}$$

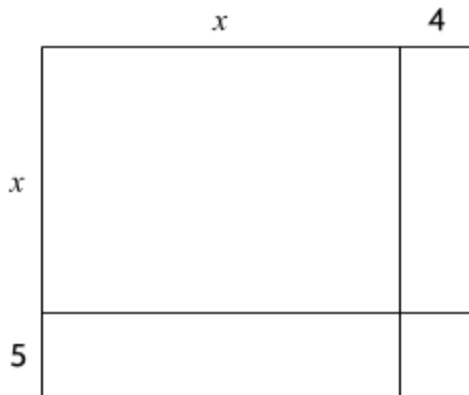
The expression  $x^2 + 6x + 3x + 18$  would usually be simplified to  $x^2 + 9x + 18$ . The example above shows that  $(x + 3)$  multiplied by  $(x + 6)$  is equal to  $x^2 + 9x + 18$ .

56) Which expression shows the area of the rectangle below?

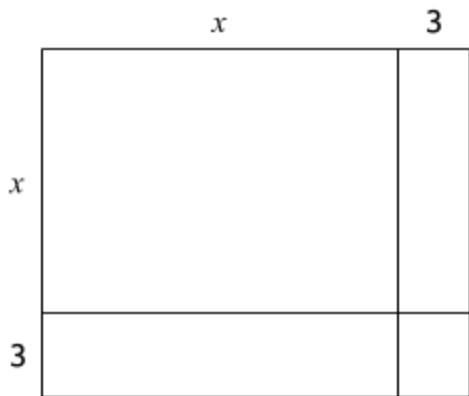
- A.  $x^2 + 3x + 4$
- B.  $x^2 + 3x$
- C.  $x^2 + 4x + 3$
- D.  $3x^2 + x$



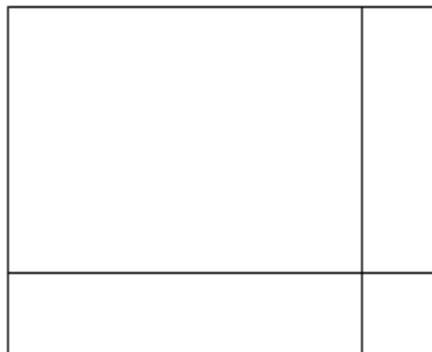
57) What expression shows the area of this rectangle?



58) What is the area of this rectangle?



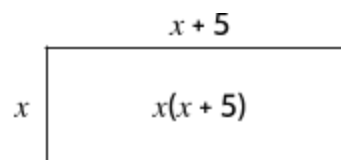
59) What  $(x + 4)$  multiplied by  $(x + 6)$ ?



Tools of Algebra: Nonlinear Functions (Part 1)

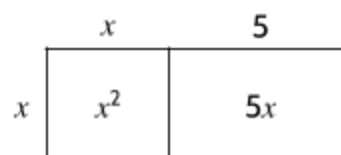
Different algebraic expressions can be equivalent to each other. The expressions  $x(x + 5)$  and  $x^2 + 5x$  are equivalent. You can see this with an area model that shows multiplication of  $x$  by  $x + 5$  in two ways:

In the first figure,  $x \cdot (x + 5)$  is equal to  $x(x + 5)$ .



In the second figure,  $x \cdot (x + 5)$  is equal to  $x^2 + 5x$ .

The area of the two rectangles is equal, so  $x(x + 5) = x^2 + 5x$ .

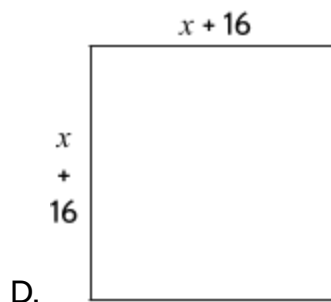
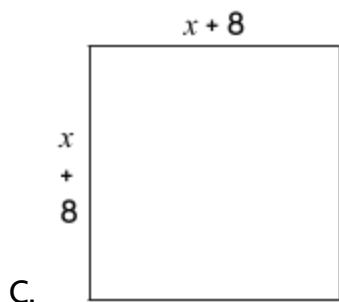
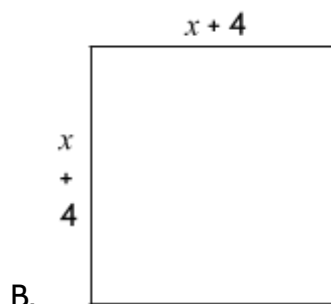
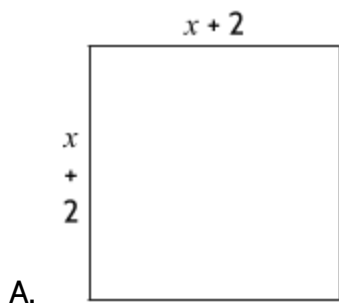


You can also use the distributive property to show that  $x(x + 5)$  and  $x^2 + 5x$  are equivalent.

$$\begin{aligned} x \cdot (x + 5) &= (x \cdot x) + (x \cdot 5) \\ &= x^2 + 5x \end{aligned}$$

The expression  $x(x + 5)$  is in its *factored* form. *Factors* are the values that are multiplied together to get another value. A *product* is the result of multiplication. In this case,  $x$  can be multiplied by  $x + 5$  to get  $x^2 + 5x$ , so  $x$  and  $x + 5$  are considered factors of  $x^2 + 5x$ , which is a product of the two factors. The expression  $x^2 + 5x$  is considered the *standard form* of these two equivalent expressions.

60) Which rectangle has an area equivalent to  $x^2 + 8x + 16$ ?



Tools of Algebra: Nonlinear Functions (Part 1)

The following expressions are in factored form. Convert each expression to standard form.

61)  $(x + 2)(x + 2)$

The expression in standard form is  $x^2 + 2x + 2x + 4$ , which can be simplified to  $x^2 + 4x + 4$ .

	$x$	$2$
$x$	$x^2$	$2x$
$2$	$2x$	$4$

62)  $(x + 5)(x + 3)$

	$x$	$3$
$x$		
$5$		

63)  $(x + 2)(x + 1)$

	$x$	$1$
$x$		
$2$		

64)  $(x + 5)(x + 1)$


65) Which expression below is equivalent to  $x^2 + 8x + 16$ ?

A.  $(x + 2)(x + 2)$

C.  $(x + 1)(x + 2)$

B.  $(x + 4)(x + 4)$

D.  $(x + 3)(x + 5)$

## Introduction to Quadratic Functions - Answer Key

1) The outputs are 1, 4, 9, 16, and 25.

2) 49

There is a mix of a mix of linear, quadratic, and exponential growth patterns below.

Continue the pattern.

Describe the pattern.

3) 18, 16, 14, 12, 10, 8, ...

Subtract 2 each time.

4) 7, 14, 21, 28, 35, 42, ...

Add 7.

5) 1, 4, 9, 16, 25, 36, ...

Add 3, 5, 7, 9, 11, etc. (2 more each time.)

6) 1, 2, 4, 8, 16, 32, ...

Double or multiply by 2.

7) 1, 3, 6, 10, 15, 21, ...

Add 2, 3, 4, 5, 6, etc. (1 more each time.)

8) 1, 3, 9, 27, 81, 243, ...

Triple or multiply by 3.

9) 3, 9, 19, 33, 51, 73, ...

Add 6, 10, 14, 18, 22, etc. (4 more each time.)

10) 3, 7, 13, 21, 31, 43, ...

Add 4, 6, 8, 10, 12, etc. (2 more each time.)

11) 3, 6, 9, 12, 15, 18, ...

Add 3.

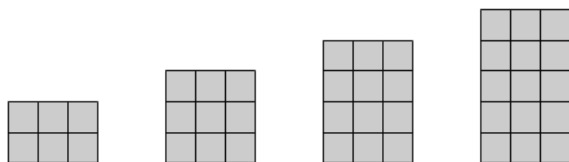
12) 2, 6, 12, 20, 30, 42, ...

Add 4, 6, 8, 10, 12, etc. (2 more each time.)

13) 16, 8, 4, 2, 1,  $\frac{1}{2}$ , ...

Divide by 2.

14) Visual Pattern A



a) Figure 1      Figure 2      Figure 3      Figure 4

b) Complete the table.

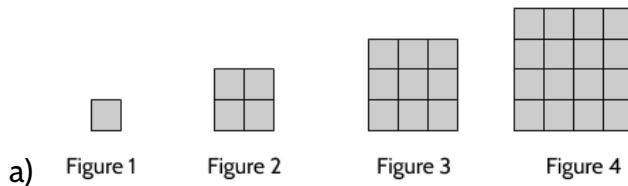
Figure	Number of Squares
1	6

*Tools of Algebra: Nonlinear Functions (Part 1)*

2	9
3	12
4	15
5	18
6	21

- c) There are different ways to see how the growth is happening:
- Three squares are added to the top of each figure.
  - Three squares are added to the bottom of each figure, pushing it up.
  - Figure 1 is  $2 \times 3$ , Figure 2 is  $3 \times 3$ , Figure 3 is  $4 \times 3$ , etc. Each figure has one more multiple of 3.

15) Visual Pattern B



- b) Complete the table.

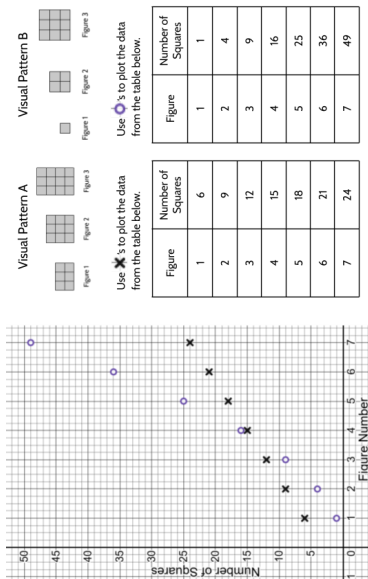
Figure	Number of Squares
1	1
2	4
3	9
4	16
5	25
6	36

- c) There are different ways to see how the growth is happening:
- An additional row and column is added in each figure.
  - The figures are growing squares.
  - Figure 1 is  $1 \times 1$ , Figure 2 is  $2 \times 2$ , Figure 3 is  $3 \times 3$ , etc.

16) Complete the table and graph the points.



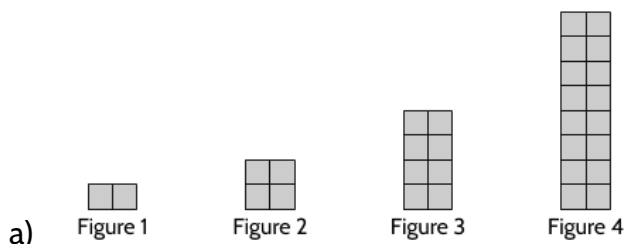
## Tools of Algebra: Nonlinear Functions (Part 1)



There are many different things to notice in the tables and graphs of the two functions. Here are a few examples:

- Visual Pattern A starts off with more squares for the first 3 figures, but Visual Pattern B has more squares starting with Figure 4.
- Visual Pattern A and B have almost the same number of squares in Figure 4, but Visual Pattern B has 1 more.
- Visual Pattern A is growing at a constant rate. Visual Pattern B is growing faster and faster.
- The graph of Visual Pattern A is line. The graph of Visual Pattern B is a curve.

### 17) Visual Pattern C



b) Complete the table.

Figure	Number of Squares
1	2
2	4
3	8

Tools of Algebra: Nonlinear Functions (Part 1)

4	16
5	32
6	64

- c) There are different ways to see how the growth is happening:
- There are 2 new squares, then 4 new squares, then 8 new squares, etc.
  - The number of squares is doubling or being multiplied by 2 in each figure.
- 18) We're interested in what you notice.
- 19) Figure A. There is a constant rate of change of 3 new squares each time.
- 20) Figure B is a quadratic function and Figure C is an exponential function. Both are nonlinear functions. We will learn more about them in the rest of the packet.
- 21) B
- 22) 7
- 23) 6
- 24) 14
- 25) D
- 26) 4.5
- 27) 11

Rule: $y = 3x$	
$x$	$y$
11	33
2	6
5	15
$\frac{1}{3}$	1
.25	.75

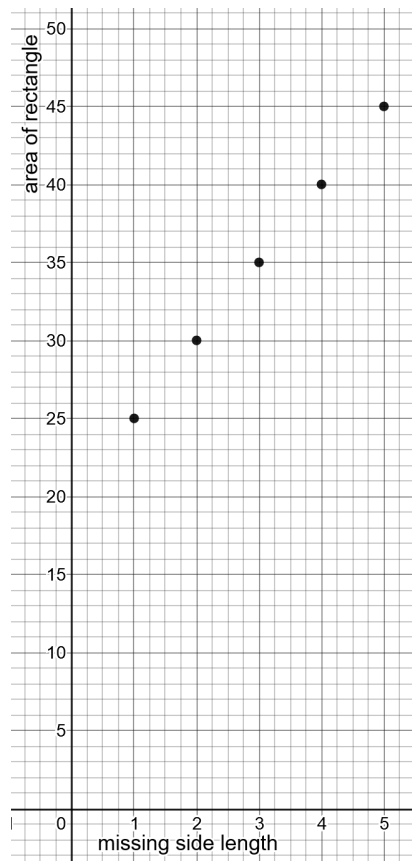
- 28)
- 29) D
- 30)  $x = 2$
- 31) There are different possible equations. Here are a few:
- $5(4 + x) = y$

Tools of Algebra: Nonlinear Functions (Part 1)

- $(5 \cdot 4) + (5 \cdot x) = y$
- $20 + 5x = y$

$x$	$y$
1	25
2	30
3	35
4	40
5	45

32)

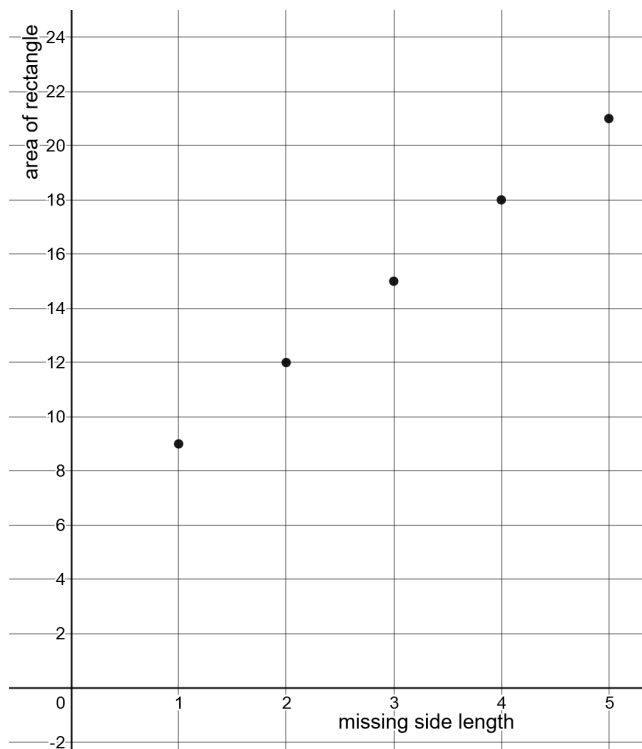


Tools of Algebra: Nonlinear Functions (Part 1)

33) Linear

34)

$x$	$y$
1	9
2	12
3	15
4	18
5	21



35) Try drawing different rectangles. The goal is to find a rectangle that has 60 squares. Is the length 4 inches longer than the width? You may have to try drawing a few different rectangles.

36) The table is another way try out different rectangles. Tables are a great problem-solving strategy because they help us organize our work so we can look back and see patterns.

37) Yes.  $(6)^2 + 4(6) = 60$ . We also know that if 6 makes  $x(x + 4) = 60$ , then it must make  $x^2 + 4x = 60$  true since the  $x(x + 4)$  and  $x^2 + 4x$  are equivalent.

38)  $6 \times 9$

39) B

40)  $8 \times 16$

41) 50

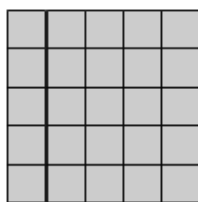


Figure 5

42) True.

43) 225

44) 56.25

45) 13

46) We will define a quadratic function later. For now, we want to know what you think.

47) There are many ways to find the area of a rectangle.

- You could count them all. That's exhausting though.
- You could count the length of the edges and multiply.
- You could split the rectangle into smaller rectangles and then find the area of those pieces. How many different ways can you do this?

48) We explain this method on the next page after the question.

49)

	10	5
10	100	50
8	80	40

5 · 8	→	40
5 · 10	→	50
10 · 8	→	80
10 · 10	→	+ 100
		270

50) There are many different ways to do this. We show one way on the next page after the question.

51)

23	
× 17	
21	(7 × 3)
140	(7 × 20)
30	(10 × 3)
200	(10 × 20)
391	

52)

16	
× 35	
30	(5 × 6)
50	(5 × 10)
180	(30 × 6)
300	(30 × 10)
560	

Tools of Algebra: Nonlinear Functions (Part 1)

$$\begin{array}{r}
 14 \\
 \times 24 \\
 \hline
 16 \quad (4 \times 4) \\
 40 \quad (4 \times 10) \\
 80 \quad (20 \times 4) \\
 200 \quad (20 \times 10) \\
 \hline
 336
 \end{array}$$

53)

$$\begin{array}{r}
 25 \\
 \times 13 \\
 \hline
 15 \quad (3 \times 5) \\
 60 \quad (3 \times 20) \\
 50 \quad (10 \times 5) \\
 200 \quad (10 \times 20) \\
 \hline
 325
 \end{array}$$

54)

55)  $(x + 3)(x + 1)$  or  $x^2 + 3x + x + 3$  or  $x^2 + 4x + 3$

56) C

57)  $(x + 5)(x + 4)$  or  $x^2 + 5x + 4x + 20$  or  $x^2 + 9x + 20$

58)  $(x + 3)(x + 3)$  or  $x^2 + 3x + 3x + 9$  or  $x^2 + 6x + 9$

59)  $(x + 4)(x + 6)$  or  $x^2 + 4x + 6x + 24$  or  $x^2 + 10x + 24$

60) B

61)  $x^2 + 4x + 4$

62)  $x^2 + 8x + 15$

63)  $x^2 + 3x + 2$

64)  $x^2 + 6x + 5$

65) B

## Three Views of a Quadratic Function

Functions show us how quantities change in relation to each other. Quadratic functions have growth patterns related to the size of different squares. All quadratics include an  $x^2$  ( $x$  “squared”) term.

We can represent functions in different ways to help us see the change in different ways. In **Tools of Algebra: Linear Functions**, we looked at three views of a function: a table, an equation, and a graph. Like linear functions, quadratic functions can be represented with each of these views. We will see differences when we compare quadratic tables, equations, and graphs with representations of linear functions.

### Tables

Let’s start by looking at some quadratic functions in a table format. Use number patterns to find the missing outputs.

- 1) Complete the table.

How did you find the missing *Out* numbers?

<i>In</i>	<i>Out</i>
1	2
2	4
3	6
4	
5	10
6	

- 2) Complete the table.

How did you find the missing *Out* numbers?

<i>In</i>	<i>Out</i>
1	3
2	6
3	11
4	18
5	
6	

- 3) What difference do you see between these two functions?

## Tools of Algebra: Nonlinear Functions (Part 1)

The first table on the previous page is for a linear function. We can tell that is a linear function because the *rate of change* (the difference in the outputs when the inputs are *consecutive*<sup>5</sup>) is constant. Since the rate of change is +2 between the outputs of 2 and 4, and between 4 and 6, we can predict that the output will be 8 for an input of 4. Each output is 2 more than the previous output.

	<i>In</i>	<i>Out</i>	
	1	2	rate of change ↓ +2 +2 +2 +2 +2
+1	2	4	
+1	3	6	
+1	4	8	
+1	5	10	
+1	6	12	

	<i>In</i>	<i>Out</i>	
	1	3	rate of change ↓ +3 +5 +7 +9 +11
+1	2	6	
+1	3	11	
+1	4	18	
+1	5	27	
+1	6	38	

The second table on the previous page is for a quadratic function. We can tell that it is not a linear function because the *rate of change* is not constant. Between the output of 3 and the output of 6, there is a rate of change of +3. Between the output of 6 and the output of 11, there is a rate of change of +5. Since the change between each output is not the same, we know the function is not growing at a constant rate of change.

- 4) Complete the outputs in these two function tables, then write the rates of change.

	<i>In</i>	<i>Out</i>	
	1	3	+6
	2	9	
	3	19	
	4	33	
	5		
	6		

	<i>In</i>	<i>Out</i>	
	1	2	+3
	2	5	
	3	9	
	4	14	
	5		
	6		

<sup>5</sup> One after the other. The numbers 3, 4, 5, 6 are in consecutive order.



## Tools of Algebra: Nonlinear Functions (Part 1)

You may see functions in vertical tables like we have used so far, or horizontal tables, as shown below.

- 5) Complete the table.

<i>In</i>	1	2	3	4	5	6
<i>Out</i>	2	6	12	20		

How did you find the missing *Out* numbers?

- 6) Complete the table.

<i>In</i>	1	2	3	4	5	6
<i>Out</i>	1	3	6		15	

How did you find the missing *Out* numbers?

Even though the rate of change in these quadratic functions isn't constant, there are patterns you can use to predict outputs for each function.

Here are the outputs from some of the quadratic functions above:

1, 4, 9, 16, 25, ...

2, 5, 9, 14, 20, ...

3, 9, 19, 33, 51, ...

2, 6, 12, 20, 30, ...

- 7) What patterns do you see in these series of numbers?

Tools of Algebra: Nonlinear Functions (Part 1)

Let's look at each function in a horizontal table. Describe the pattern you see.

$x$	1	2	3	4	5	6
$y$	1	4	9	16	25	36

8)

+3                      +5                      +7                      +9                      +11

Description: The rate of change is +3, +5, +7, etc. The differences between outputs are going up by 2.

$x$	1	2	3	4	5	6
$y$	3	9	19	33	51	73

9)

+6                      +10                      +14                      +18                      +22

Description:

$x$	1	2	3	4	5	6
$y$	2	5	9	14	20	27

10)

+3                      +4                      +5                      +6                      +7

Description:

$x$	1	2	3	4	5	6
$y$	2	6	12	20	30	42

11)

+4                      +6                      +8                      +10                      +12

Description:

12) How is the function below different from the four functions above?

$x$	1	2	3	4	5	6
$y$	5	10	15	20	25	30

### Tools of Algebra: Nonlinear Functions (Part 1)

If you look at the difference in the outputs of the function above, you will see that the rate of change stays the same between each output. Since the rate of change is constant, the function below is linear.

$x$	1	2	3	4	5	6
$y$	5	10	15	20	25	30

+ 5                  + 5                  + 5                  + 5                  + 5

On the other hand, the other four functions on the previous page have a rate of change that is changing. The changing difference between the outputs lets us know that these are nonlinear functions. For example, the function below is nonlinear.

$x$	1	2	3	4	5	6
$y$	2	6	12	20	30	42

+ 4                  + 6                  + 8                  + 10                  + 12

One interesting thing you might notice is that, even though the differences between the outputs don't stay the same, the differences between outputs are going up at a constant rate.

$x$	1	2	3	4	5	6
$y$	2	6	12	20	30	42

+ 4                  + 2                  + 6                  + 2                  + 8                  + 2                  + 10                  + 2                  + 12

In this function, the differences between outputs are going up 2 between each input. In other words, the rate of change is changing at a constant rate of +2.

13) Which of these two tables represents a nonlinear function?

A. 

$x$	1	2	3	4	5	6
$y$	12	24	36	48	60	72

B. 

$x$	1	2	3	4	5	6
$y$	12	20	30	42	56	72

## Rules and Equations

A function is a relationship between an input and output. The inputs are the numbers that are put into the function. The function does something to the input and produces the output number. One way to look at a function is by understanding its rule or equation. The function rule determines what the function does to the input.

Here is an example of a function rule written as a number sentence:

Multiply the input by 3 more than the input.

Any number that is put into the function is multiplied by a number that is 3 more than the original number:

$$0 \cdot 3 = 0$$

$$1 \cdot 4 = 4$$

$$2 \cdot 5 = 10$$

$$3 \cdot 6 = 1 \cdot 8$$

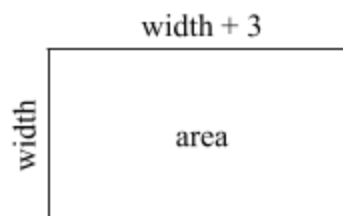
$$4 \cdot 7 = 2 \cdot 8$$

Etc.

This same rule can be shown with the area of a rectangle. To calculate the area, we multiply the width by the length:

$$\text{width} \cdot (\text{width} + 3) = \text{area}$$

The area of the rectangle is a function of the width of the rectangle. As the width changes, so does the area. If the width is 1, then the area is 4. If the width is 2, then the area is 10. Each input of the width produces an output of the area.



- 14) Complete the function table based on the function rule.

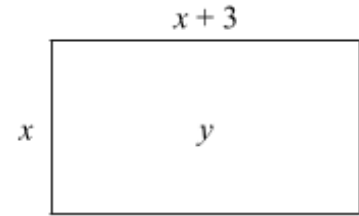
Rule: Multiply the input by 3 more than the input.

<i>Input</i>	<i>Input</i> + 3	<i>Input</i> · ( <i>Input</i> + 3)	<i>Output</i>
5	5 + 3	5(5 + 3)	40
9	9 + 3	9(9 + 3)	
20			
6			

## Tools of Algebra: Nonlinear Functions (Part 1)

We can also use variables to show this rule in an equation. The rectangle on the right shows the relationship between *width* and *area*. Instead of using *width*, we now use the variable  $x$ , and instead of *area*, we use  $y$ .

$$x(x + 3) = y$$



In this equation, the area of the rectangle ( $y$ ) is a function of the width of the rectangle ( $x$ ). As the value of  $x$  changes, so does the value of  $y$ . Each input of  $x$  produces an output of  $y$ .

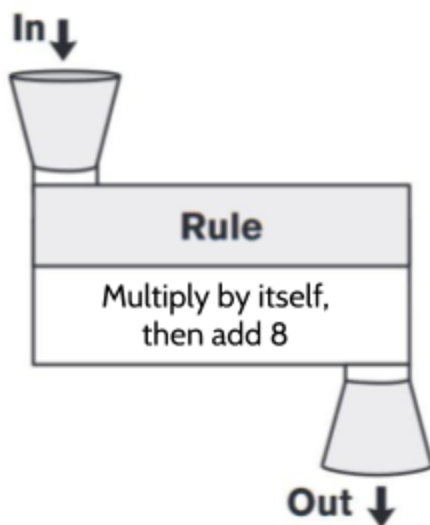
In the rest of this section, we will practice using function rules and equations with variables to complete function tables.

---

In the following function machine, the function rule is “*Multiply by itself, then add 8.*” Following the rule, when the number 0 placed in the machine, the rule is applied to the input number, which is multiplied by itself, then added to 8. The number 8 comes out of the function machine.

In the following pages, we will use function machines and keep track of our calculations with function tables. We add a *Calculation* column in each table to show how we find the output number.

- 15) Complete the table for each of the remaining inputs.



In	Calculation	Out
0	$0 \cdot 0 + 8$	8
1	$1 \cdot 1 + 8$	
2		
3		
4		
5		

## Tools of Algebra: Nonlinear Functions (Part 1)

As we know, the function rule can also be written as an equation. The word *equation* comes from the word *equal*. In mathematics, an equation shows that the expression on one side of the equals sign is equal in value to the expression on the other side. Here is an example of the previous function rule written as an equation:

$$Out = In \cdot In + 8$$

This can be read as, “The output is equal to the input multiplied by itself and added to 8.” In function equations, the output is usually on the left side of the equation and the function rule, including the input, is on the right side of the equation.

The function equation can also be written with the variables  $x$  and  $y$ :

$$y = x \cdot x + 8$$

In the equation,  $x$  is the input number and  $y$  is the output number.

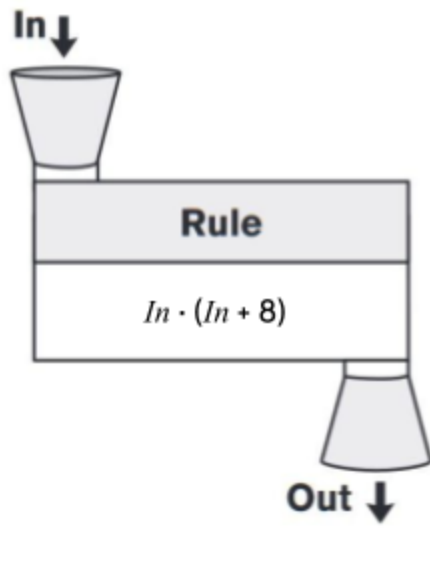
You will also sometimes see the output written as  $f(x)$ . This can be read as “the function of  $x$ ” or “ $f$  of  $x$ ” and means the same thing as  $y$ . Both  $y$  and  $f(x)$  refer to the output of the function.

There are many ways of writing function rules and equations. Each of the following equations mean the same thing:

Input	Output	Equations
$In$	$Out$	$Out = In \cdot In + 8$
$x$	$y$	$y = x \cdot x + 8$
$x$	$f(x)$	$f(x) = x^2 + 8$
$n$	$f(n)$	$f(n) = n^2 + 8$

- 16) One version of this function uses the expression  $In \cdot In + 8$ . How is this different from the expression  $In \cdot (In + 8)$ ?

- 17) Using the function rule, complete the table for each of the remaining inputs.

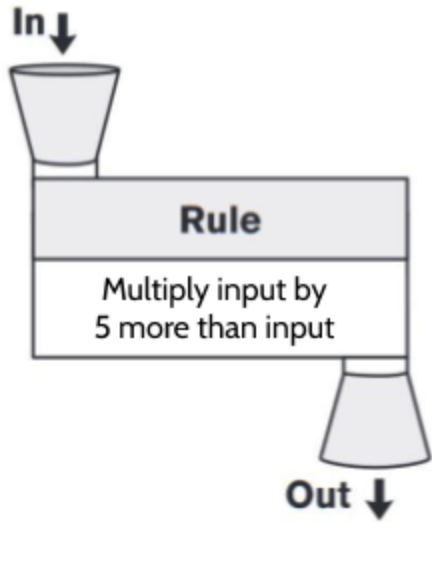


In	Calculation	Out
0	$0 \cdot (0 + 8)$	0
1	$1 \cdot (1 + 8)$	
2		
3		
4		
5		

- 18) Compare the outputs from  $Out = In \cdot (In + 8)$  and  $Out = In \cdot In + 8$ .  
What do you notice?
- 19) Which of these rules is equivalent to the expression in the function machine above?
- Multiply input by 8
  - Multiply input by input, then add 8
  - Multiply input by 8 more than input
  - Multiply input by input
- 20) If 10 is put into the function machine, what number comes out?
- 21) If 240 comes out of the function machine, what number went in?

22) Translate the following rule into an expression: *Multiply input by 5 more than input.*

23) Using the function rule, complete the table for each of the remaining inputs.



In	Calculation	Out
0	$0 \cdot (0 + 5)$	0
1	$1 \cdot (1 + 5)$	
2		
3		
4		
5		

24) Which of the following equations is not equivalent to the rule above?

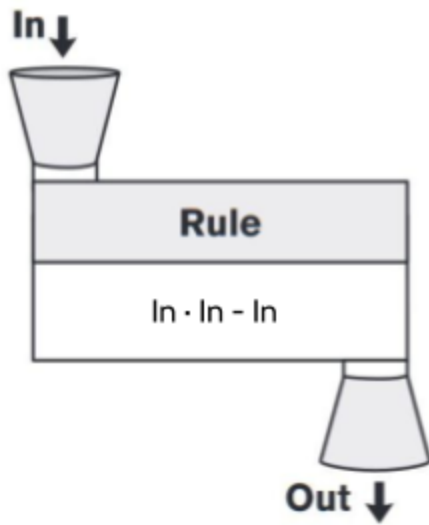
- A.  $Out = In \cdot In + 5$
- B.  $y = x(x + 5)$
- C.  $f(x) = x(x + 5)$
- D.  $f(n) = n^2 + 5n$

25) In the table above, if the input was 10, what is the output?

26) If the output is 84, what was the input?

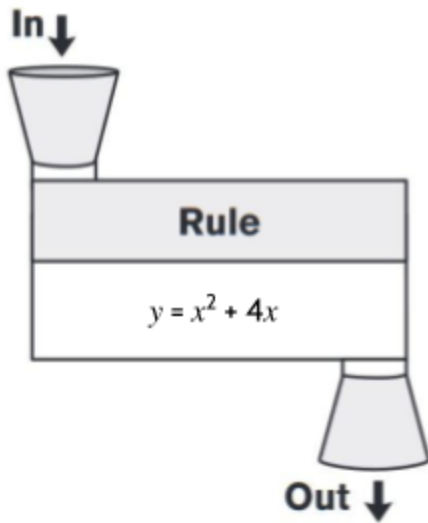


27) Complete the table.



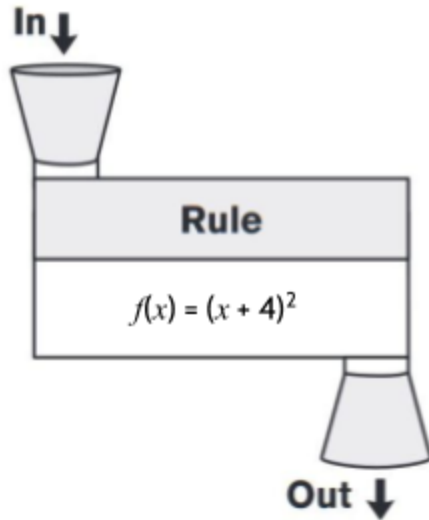
In	Calculation	Out
5	$5 \cdot 5 - 5$	20
8	$8 \cdot 8 - 8$	
1		
3		
		2
10		

28) Complete the table.



$x$	Calculation	$y$
3	$(3 \cdot 3) + (4 \cdot 3)$	21
5		
2	$(2 \cdot 2) + (4 \cdot 2)$	
8		
	$(7 \cdot 7) + (4 \cdot 7)$	
		32

29) Complete the table.



$x$	Calculation	$f(x)$
7	$(7 + 4)^2$	121
5	$(5 + 4)^2$	
6		
		16
3		
10	$(10 + 4)^2$	

30)  $f(7)$  is 121 because when 7 goes into this function, 121 comes out. What is  $f(12)$ ?

31) If  $f(x)$  is 25, what is  $x$ ?

32) Which of the following equations is equivalent to the rule above?

- A.  $f(x) = 2x + 8$
- B.  $f(x) = x^2 + 4$
- C.  $f(x) = x^2 + 16$
- D.  $f(x) = x^2 + 8x + 16$

Tools of Algebra: Nonlinear Functions (Part 1)

33) Complete the function tables for these two functions.

Rule: $y = (x + 2)(x + 3)$		
$x$	Calculation	$y$
-5	$(-5 + 2)(-5 + 3)$	6
-4		
-3		0
-2		
-1		
0	$(0 + 2)(0 + 3)$	
1		
2		
3		
4		
5		

Rule: $y = x^2 + 5x + 6$		
$x$	Calculation	$y$
-5	$(-5)^2 + 5(-5) + 6$	
-4	$(-4)^2 + 5(-4) + 6$	2
-3		
-2		
-1		
0		
1		
2		
3		
4		
5		56

What do you notice?

## Tools of Algebra: Nonlinear Functions (Part 1)

You may have noticed that the outputs for  $y = (x + 2)(x + 3)$  and  $y = x^2 + 5x + 6$  are the same. This is because the two functions are equivalent. We can see this by multiplying  $(x + 2)$  and  $(x + 3)$ :

$$\begin{aligned} (x + 2) \cdot (x + 3) &= (x \cdot x) + (x \cdot 3) + (2 \cdot x) + (2 \cdot 3) \\ &= x^2 + 5x + 6 \end{aligned}$$

We can also use an area model to show how  $(x + 2)$  multiplied by  $(x + 3)$  is equal to  $x^2 + 5x + 6$ :

	$x$	$3$
$x$	$x^2$	$3x$
$2$	$2x$	$6$

$$\begin{array}{r} 6 \\ 3x \\ 2x \\ +x^2 \\ \hline x^2 + 2x + 3x + 6 \\ x^2 + 5x + 6 \end{array}$$

The equation  $y = x^2 + 5x + 6$  is in the standard form of this quadratic function.  $y = (x + 2)(x + 3)$  is in the factored form of the same function.

34) Which of the following equations is equivalent to  $y = (x + 6)(x + 6)$ ?

- A.  $y = x^2 + 36x + 12$
- B.  $y = x^2 + 12x + 36$
- C.  $y = x^2 + 12x$
- D.  $y = x^2 + 36x$

35) Which of the following equations is equivalent to  $y = x^2 + 10x + 21$ ?

- A.  $y = (x + 3)(x + 7)$
- B.  $y = (x + 2)(x + 5)$
- C.  $y = (x + 1)(x + 10)$
- D.  $y = (x + 5)(x + 5)$

### Connecting Equations and the Function Table

By analyzing equations, we can predict some features of the function table. For example, let's review how we have been able to use the equation of a linear function to predict the rate of change and starting amount in the function table:

In a linear function, the coefficient of  $x$  (the number multiplied by  $x$ ) tells us the rate of change of the function. In this table, we can see that the rate of change is 3. We see it multiplied by  $x$  in the equation rule and also in the difference between outputs in the function table.

Rule: $y = 3x + 5$	
$x$	$y$
0	5
1	8
2	11
3	14
4	17
5	20

Diagram annotations: On the left side of the table, five brackets labeled '+1' indicate the constant increase in the x-values from one row to the next. On the right side, five brackets labeled '+3' indicate the constant increase in the y-values from one row to the next.

The starting amount (value of  $y$  when  $x$  is 0) can be seen in the equation and in the table. If  $x$  is 0, then  $y$  is 5. The + 5 at the end of the equation is connected to the 5 next to the 0 in the function table.

When a linear function is shown on a graph, the rate of change is called *slope* and the starting amount is called the *y-intercept*.

Look at the function equation  $y = 2x + 10$ .

- 36) What is the rate of change (slope) of this function?
- 37) What is the starting amount (rate of change) of this function?

As we have seen, quadratic functions don't have a constant rate of change, so we won't be able to find the rate of change in the equation. However, there are other features of the function table that we can predict by looking at the function equation.

- 38) For the function  $y = x^2 + 5x + 6$ , what is the value of  $y$  when  $x$  is 0?

Tools of Algebra: Nonlinear Functions (Part 1)

Let's look at  $y = x^2 + 5x + 6$  for  $x = 0$ .

$$y = 0^2 + 5(0) + 6$$

$$y = 6$$

The value of  $y$  is 6 when  $x$  is 0. That is the definition of starting amount. If you look at the function table for  $y = x^2 + 5x + 6$  on the right, you can see the starting amount in the table there as well. When  $x$  is 0,  $y$  is 6.

We can use the function equation in standard form to predict the value of  $y$  when  $x$  is 0 (the starting amount).

39) Looking at the table, what do you notice when  $y$  is 0?

Rule (Standard Form): $y = x^2 + 5x + 6$	
$x$	$y$
-5	6
-4	2
-3	0
-2	0
-1	2
0	6
1	12
2	20
3	30
4	42
5	56

When  $y$  is 0,  $x$  has two possible values. It is either -3 or -2. Let's look at the equation calculations to see why there are two possible  $x$  values for  $y = 0$ .

When  $x$  is -3:

$$y = (x + 2)(x + 3)$$

$$y = ((-3) + 2) \cdot ((-3) + 3)$$

$$y = (-1) \cdot (0)$$

$$y = 0$$

When  $x$  is -2:

$$y = (x + 2)(x + 3)$$

$$y = ((-2) + 2) \cdot ((-2) + 3)$$

$$y = (0) \cdot (1)$$

$$y = 0$$

The  $x$ -values -3 and -2 are called the *roots* of the function  $y = x^2 + 5x + 6$ . A *root* is a value you can put into a function as an input for  $x$  so that 0 comes out as an output for  $y$ .

40) The function in standard form  $y = x^2 + 5x + 6$  is equivalent to  $y = (x + 2)(x + 3)$ . Looking at this form of the function equation, what do you notice about the roots?

*Tools of Algebra: Nonlinear Functions (Part 1)*

The *roots* are values you can use for  $x$  so the value of  $y$  is 0. The word *zero* is sometimes used to mean the same thing as *root*. The roots/zeros are the solutions to a function equation. If you are asked to solve a quadratic equation, this usually means that you should find the values of  $x$  that will produce a  $y$ -value of 0. One thing to notice is that quadratic equations will often have 2 solutions!

You may have noticed that the equation  $y = (x + 2)(x + 3)$  includes the numbers 2 and 3, the positive versions of -2 and -3, which are the solutions (or roots) to the function. Do you think this happens with other function equations?

41) Complete the table.

42) What are the zeros (roots) for  $y = (x + 5)(x - 3)$ ?

43) How could we have predicted the zeros for this function before completing the table?

44) What is the starting amount for this function?

Rule: $y = (x + 5)(x - 3)$		
$x$	Calculation	$y$
-5	$((-5) + 5) \cdot ((-5) - 3)$ $(0) \cdot (-8)$	0
-4	$((-4) + 5) \cdot ((-4) - 3)$ $(1) \cdot (-7)$	
-3		
-2		
-1		
0	$(0 + 5) \cdot (0 - 3)$ $(5) \cdot (-3)$	
1		
2		
3		
4		
5	$(5 + 5) \cdot (5 - 3)$ $(10) \cdot (2)$	

Tools of Algebra: Nonlinear Functions (Part 1)

45) What is the standard form of  $y = (x + 5)(x - 3)$ ?


The solutions (or zeros) are the  $x$ -values when the  $y$ -values are 0. In the last two functions, we found those numbers in the factored form of the function equation. The starting amount is the  $y$ -value when the  $x$ -value is 0. We found the starting amount in the standard form of the function equation.

Factored Form	Solutions	Standard Form	Starting Amount
$y = (x + 2)(x + 3)$	-2 and -3	$y = x^2 + 5x + 6$	6
$y = (x + 5)(x - 3)$	-5 and 3	$y = x^2 + 2x - 15$	-15

Notice that the factored form of the equation shows the solutions, but the numbers are opposite. For example, the factor  $(x + 2)$  tells us that one of the solutions is -2. Similarly, the factor  $(x - 3)$  reveals that one of the solutions is 3. The number in the factored form is the opposite of the solution. If the number in the factored form is positive, the solution is negative. If the number in the factored form is negative, the solution is positive.

Let's see why this is true by looking at another function in factored form:  $y = (x - 2)(x + 7)$ . We could find the zeros with a function table. Or we could use what we have noticed about opposite numbers in the function equation. Instead of those two methods, let's try another way.

Let's start by looking at the factored form of the equation:  $y = (x - 2)(x + 7)$ . On the right side of the equation, we have  $(x - 2)$  multiplied by  $(x + 7)$ . The values of each of these expressions will change depending on the value of  $x$ , but we know that we want the product of the two expressions to be 0. This is because we need the right side of the equation to equal the left side of the equation when  $y = 0$ . So, when  $y$  is 0,  $(x - 2)$  multiplied by  $(x + 7)$  has to be 0 as well. In other words, the product of  $(x - 2)$  and  $(x + 7)$  must be 0.



Tools of Algebra: Nonlinear Functions (Part 1)

- 46) Let's think about multiplying to get a product of 0. Fill in the blanks to complete the following equations.

$5 \times 0 = \underline{\hspace{2cm}}$

$8 \times \underline{\hspace{2cm}} = 0$

$0 \times 0 = \underline{\hspace{2cm}}$

$0 \times 25 = \underline{\hspace{2cm}}$

$\underline{\hspace{2cm}} \times \frac{1}{4} = 0$

$1,000 \times \underline{\hspace{2cm}} = 0$

- 47) What do you notice when you look at the equations above?

You may notice that each equation above includes the number 0. And they are all equal to 0. If the product of two numbers is 0, at least one of the factors has to be 0. This is called the *zero-product property*: If  $a \cdot b = 0$ , then  $a$  or  $b$  has to be equal to 0.

We can use this fact to help us solve quadratic equations!

Let's look at the quadratic function  $y = (x - 2)(x + 7)$  again. If  $y$  is 0, then either  $(x - 2)$  or  $(x + 7)$  has to be 0 as well.

$$0 = (x - 2) \cdot (x + 7)$$

One of these two expressions  
has to be equal to 0.

- 48) If  $(x - 2)$  is equal to 0, that means some number minus 2 is equal to 0.  
What number is that?

- 49) What number plus 7 is equal to 0?

*Tools of Algebra: Nonlinear Functions (Part 1)*

To find the solutions of  $y = (x - 2)(x + 7)$ , we could also set each factor equal to 0 and then solve for  $x$ :

$$x - 2 = 0$$

$$x + 7 = 0$$

$$x - 2 + 2 = 0 + 2$$

$$x + 7 - 7 = 0 - 7$$

$$x = 2$$

$$x = -7$$

Use any method you like to find the solutions for the following functions.

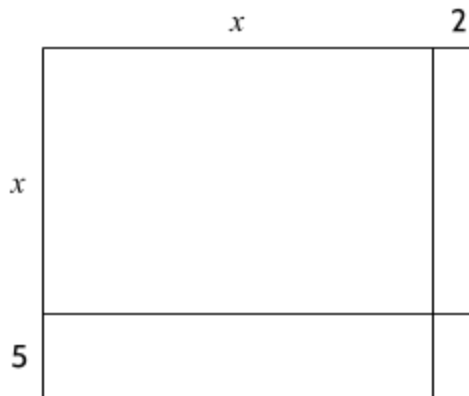
50)  $y = (x + 4)(x - 6)$

51)  $y = (x + 3)(x - 3)$

52)

$x$	0	1	2	3	4	5
$y$	8	3	0	-1	0	3

53)



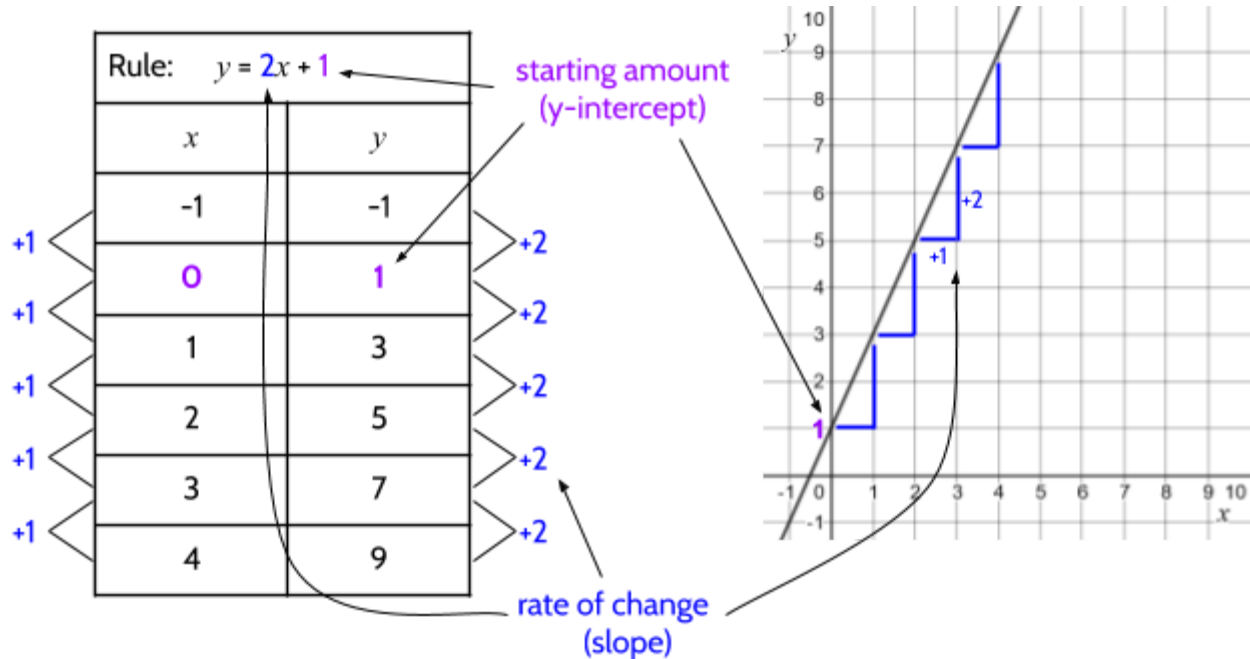
54)  $y = (x + 3)(x + 3)$

## Tools of Algebra: Nonlinear Functions (Part 1)

As we looked at earlier, we can analyze the equation of a linear function to make predictions about what the function table and the graph of the line will look like. For example, we can look at the linear function  $y = 2x + 1$  and see two things at a glance:

- The starting amount (y-intercept) is 1.
- The rate of change (slope) is 2.

We can also look at the table and the graph of the linear function to see these same features:



A common form of a linear equation is  $y = mx + b$ , where  $m$  is the rate of change or slope and  $b$  is the starting amount or y-intercept. In the linear function above,  $m = 2$  and  $b = 1$ . We can look at the value of  $m$  in order to know the rate of change in a linear function, or the slope of the line, and the value of  $b$  tells us the starting amount, or where the line crosses the  $y$  axis.

Quadratic functions in standard form are written as  $y = ax^2 + bx + c$  where  $b$  and  $c$  can be any number and  $a$  can be another number except 0.

55) The function  $y = \frac{1}{2}x^2 + 5x + 12$  is written in standard form. What is the value of  $a$ ?

*Tools of Algebra: Nonlinear Functions (Part 1)*

- 56) The function  $y = x^2 + x - 6$  is written in standard form. What is the value of  $c$ ?
- 57) For the function  $y = x^2 + 4x + c$  where  $c = 3$ , what is the value of  $y$  when  $x = 0$ ?
- 58) For the function  $f(x) = x^2 + bx + 15$  where  $b = 8$ , what is  $f(1)$ ?  
Note:  $f(1)$  means the value of the output when the input is 1.
- 59) For the function  $f(x) = x^2 + bx - 1$  where  $b = 0$ , what is  $f(4)$ ?
- 60) What are the roots of the function  $f(x) = (x + 1)(x - 1)$ ?
- 61) Consider the standard form of the quadratic function  $y = ax^2 + bx + c$ . Why do you think the value of  $a$  can't be 0? (Note: Think about the function  $y = 2x^2 + 6x + 4$ . What would happen if you replaced the 2 with 0?)

In the next section on graphing quadratic equations, we will use quadratic equations in the standard and factored forms to make predictions about the graph of the function.

## Graphs

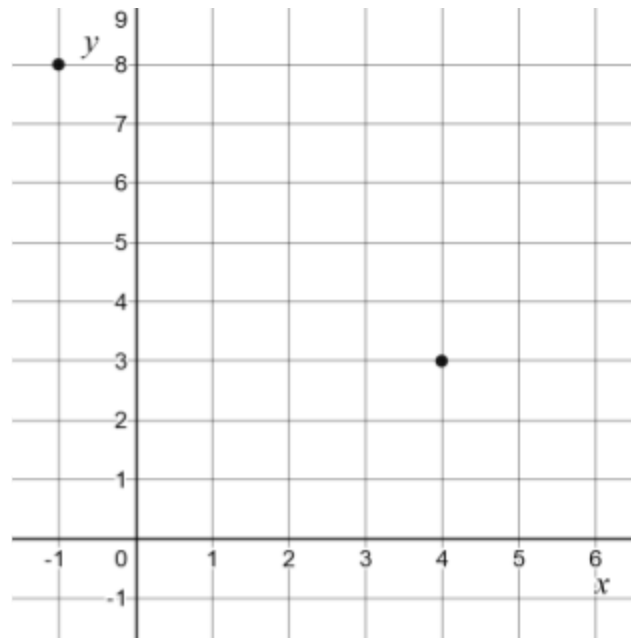
Along with equations and tables, graphs are an important way to understand functions, including quadratics. In this section, we will explore the graphs of quadratic functions.

The graph of a linear function is a line, but what does the graph of a quadratic function look like? Let's start by plotting the data from a function to create a graph.

- 62) Complete the function table for  $y = x^2 - 4x + 3$  and plot the rest of the points on the graph. Two points have been plotted for you.

Note: The ordered pair shows the location of a point on a graph. The first number is the  $x$  value and the second number is the  $y$  value.

Equation: $y = x^2 - 4x + 3$		
$x$	$y$	Ordered Pair
-1	8	(-1, 8)
0		
1		
2	-1	
3		
4	3	(4, 3)
5		

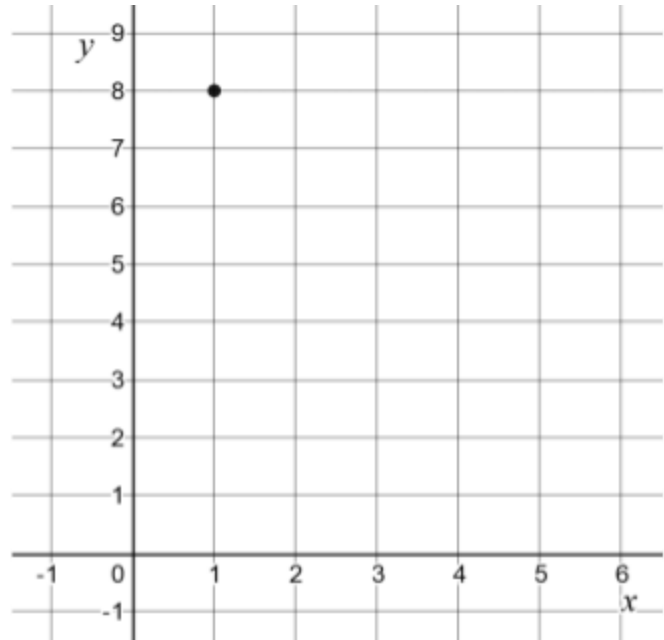


- 63) Try connecting the dots with lines. What does the graph of the function look like to you?

Tools of Algebra: Nonlinear Functions (Part 1)

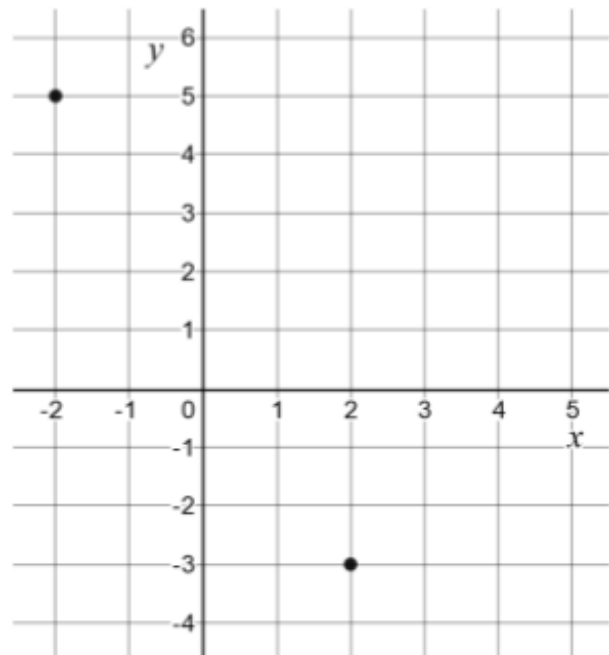
64) Complete the table for  $y = -x^2 + 4x + 5$  and plot the points on the graph.

Rule: $y = -x^2 + 4x + 5$		
$x$	$y$	Ordered Pair
-1		$(-1, 0)$
0		
1		
2		
3		
4	5	
5		



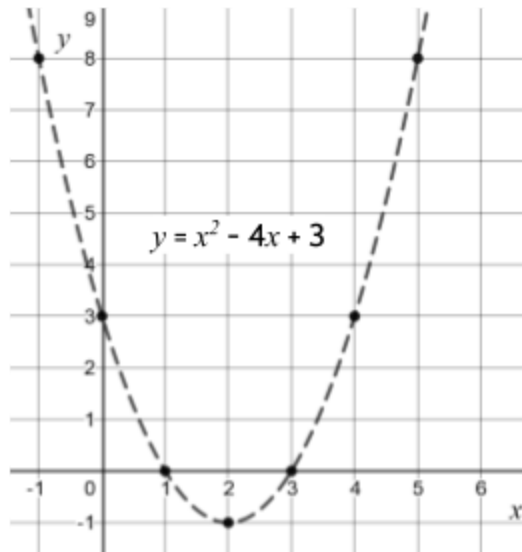
65) Complete the table for  $y = x^2 - 2x - 3$  and plot the points on the graph.

Rule: $y = x^2 - 2x - 3$		
$x$	$y$	Ordered Pair
-2	5	
-1		
0		$(0, -3)$
1		
2		
3		
4		

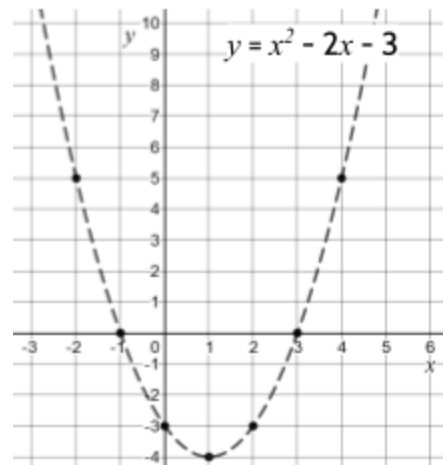
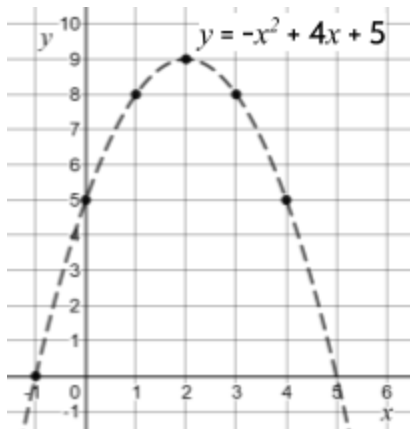


Tools of Algebra: Nonlinear Functions (Part 1)

Your graph of  $y = x^2 - 4x + 3$  might look something like this.



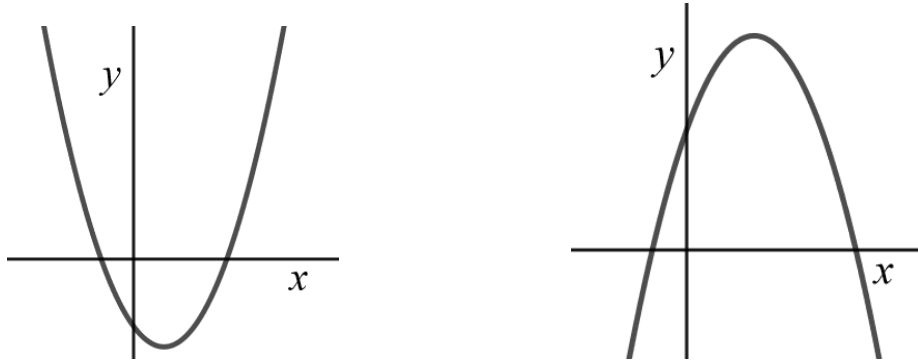
Here are graphs of the other two quadratic functions above.



66) What do you notice about the graphs of these quadratic functions?

## Tools of Algebra: Nonlinear Functions (Part 1)

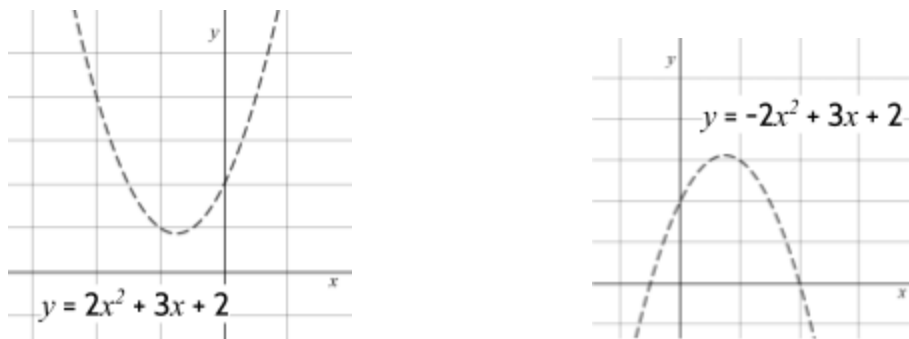
The curved shapes in the graphs are called parabolas. Every quadratic function looks like a parabola when it is graphed on the coordinate grid.



Sometimes parabolas open up like the one on the left, and sometimes they open down like the one on the right.

Imagine someone throwing a ball in the air. Wouldn't it look like the parabola on the right? It turns out that that is exactly the path a projectile takes if it is shot through the air. Balls, arrows, and rockets all follow the path of a parabola. This is because of the way gravity pulls things back to Earth. The scientist Galileo figured this out 400 years ago. This is why we can use quadratic equations to calculate how objects move through the air. We'll come back to this later in the packet.

Take a look at the equations  $y = 2x^2 + 3x + 2$  and  $y = -2x^2 + 3x + 2$  and their graphs.



67) What difference do you notice between the two equations?

68) What differences do you notice between the two graphs?

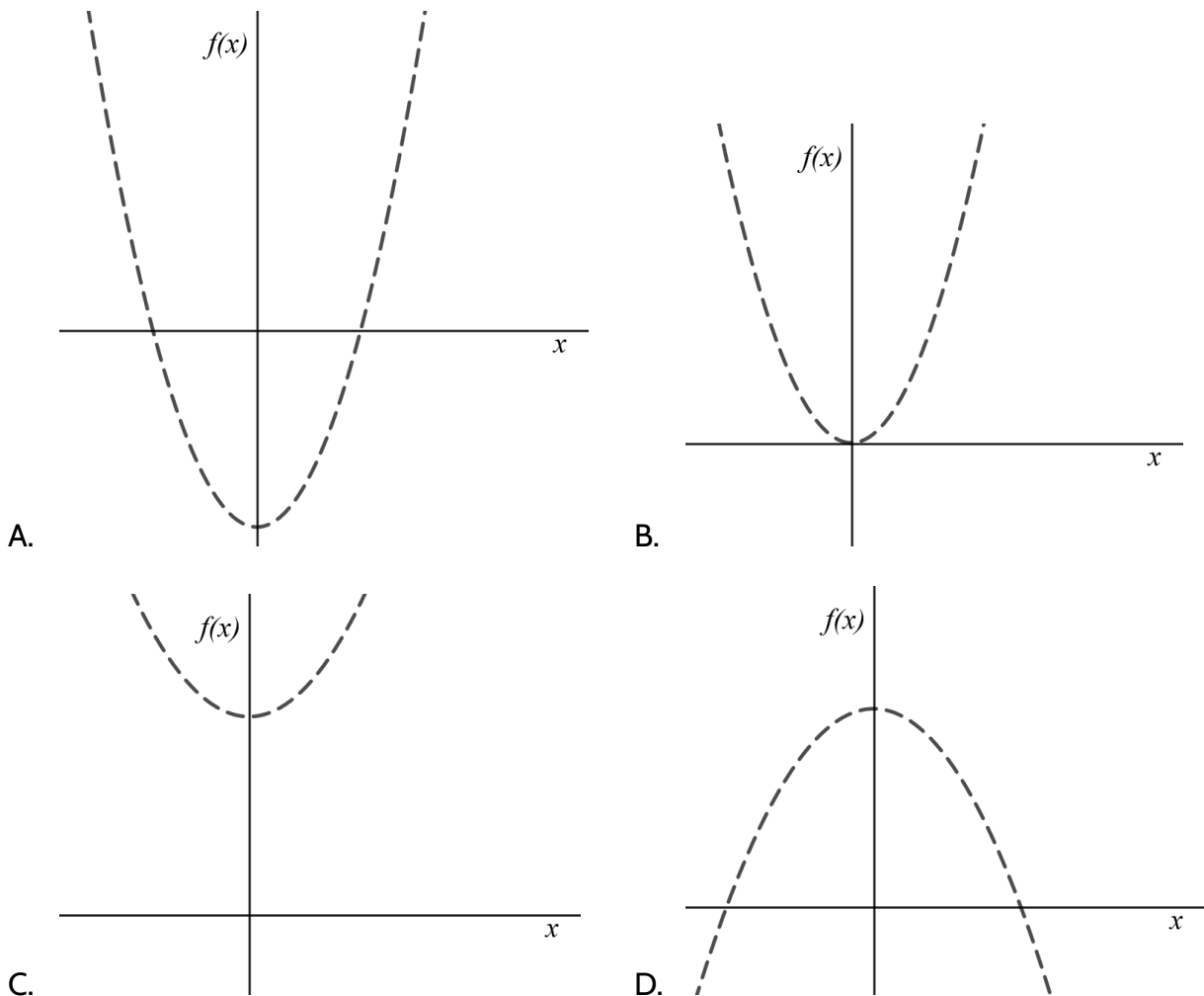


### Tools of Algebra: Nonlinear Functions (Part 1)

You may have noticed that the function  $y = -2x^2 + 3x + 2$  has a negative 2 as the coefficient for  $x^2$  and the graph is a parabola opening down. Remember that quadratics can be written in the form  $y = ax^2 + bx + c$ . If  $a$  is a negative number, the graph will be a parabola opening down with the rounded part on the top.

On the other hand, the function  $y = 2x^2 + 3x + 2$  has a positive value for  $a$ , which means that the graph is a parabola opening up, with the rounded part on the bottom.

69) Which graph matches the equation  $f(x) = -x^2 + 4$ ?

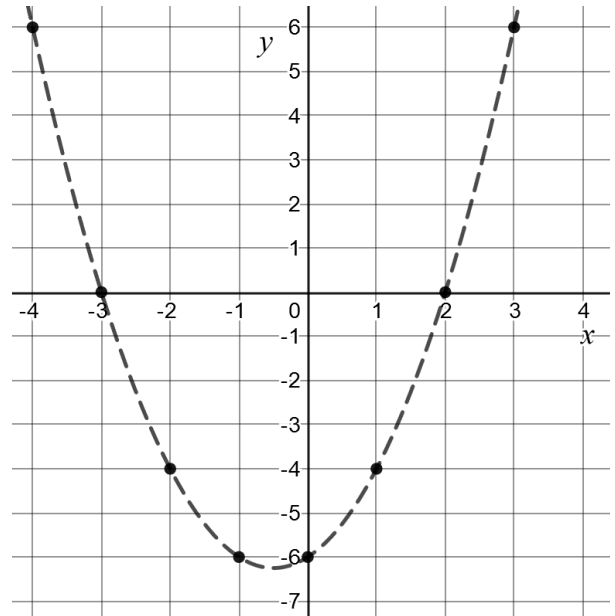


## Tools of Algebra: Nonlinear Functions (Part 1)

In the previous sections on equations, we looked at the zeros of a quadratic function, also known as *solutions* or *roots*. We noticed that the zeros of a function can be seen in the function table and in the factored form of the quadratic equation. We can also see the zeros in the graph.

70) Use the points on the graph to complete the function table below.

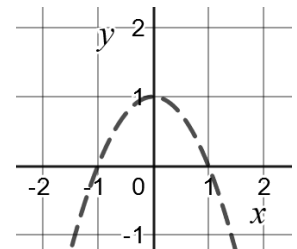
$x$	$y$
-4	6
-3	
-2	
-1	
0	
1	
2	
3	



71) What are the roots of the function shown in the graph? Remember that the roots of a function are the  $x$ -values when  $y = 0$ .

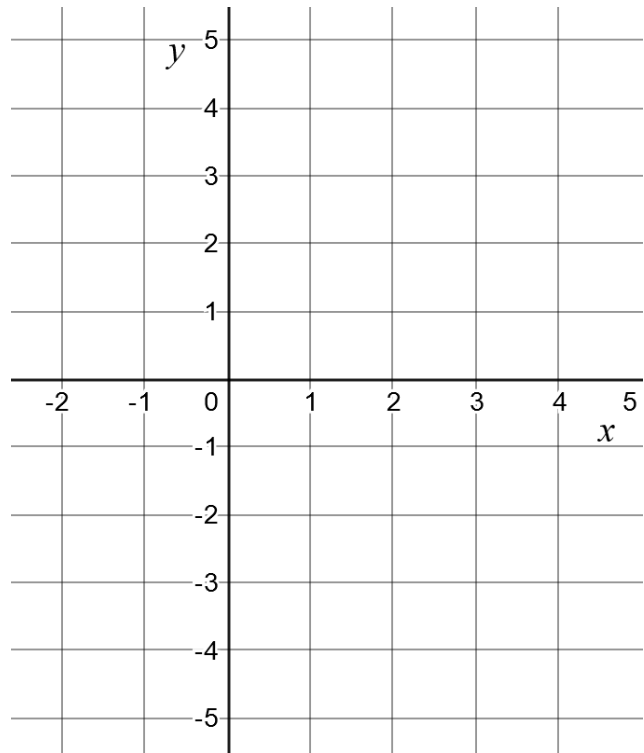
72) Circle the roots of the function on the graph. These are the two points on the  $x$  axis that intersect with the graph.

73) What are the zeros of the function  $y = (x + 1)(x - 1)$  shown in the graph?



Tools of Algebra: Nonlinear Functions (Part 1)

- 74) Looking at the function  $y = (x - 3)(x + 1)$ , make a prediction where the zeros of the function will be on the graph. Draw a circle at each of the two zeros.



- 75) Complete a function table for  $y = (x - 3)(x + 1)$  to find the roots. Are they the same as the ones you placed in the graph?

$x$	-2	-1	0	1	2	3	4
$y$							

- 76) Set each of the two factors  $(x - 3)$  and  $(x + 1)$  equal to 0 and solve for  $x$  to see if you get the same solutions.

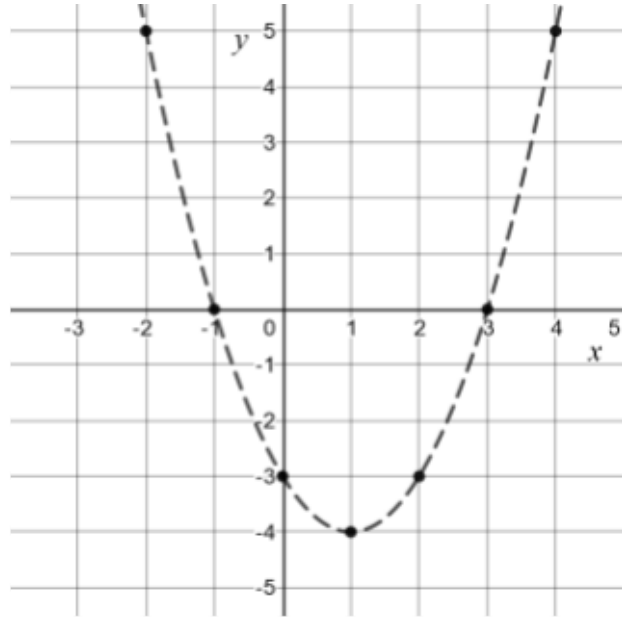
$$x - 3 = 0$$

$$x + 1 = 0$$

## Tools of Algebra: Nonlinear Functions (Part 1)

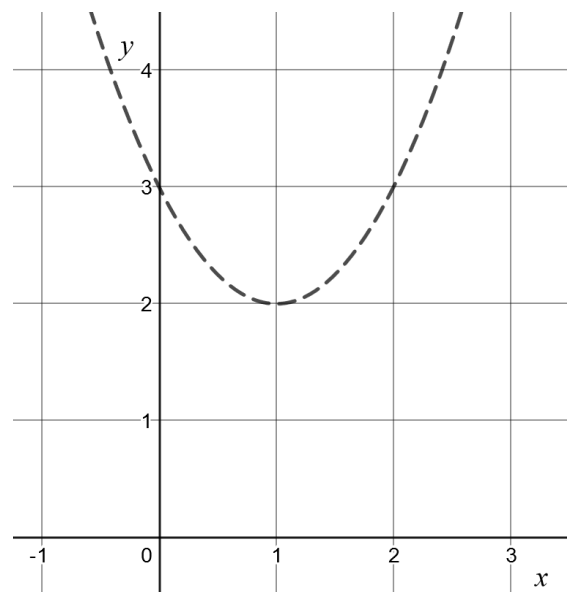
The function table and graph for  $y = (x - 3)(x + 1)$  is below. Notice that the parabola crosses the  $x$  axis at  $-1$  and  $3$ . This brings us to another name:  $x$ -intercepts. Amazingly, this is another name that means the same thing as *roots*, *solutions*, and *zeros*. The  $x$ -intercepts are the values of  $x$  when the graph intersects with the  $x$ -axis. In this function, the  $x$ -intercepts are  $-1$  and  $3$ .

Equation: $y = (x - 3)(x + 1)$	
$x$	$y$
$-2$	$5$
$-1$	$0$
$0$	$-3$
$1$	$-4$
$2$	$-3$
$3$	$0$
$4$	$5$



There is another feature of quadratic functions that you may have noticed. In some graphs, like the one above, there is a minimum value for  $y$ . In the function  $y = (x - 3)(x + 1)$ , the minimum is  $-4$ . You can see this in the function table and in the graph. The  $y$ -value of  $-4$  happens when the input is  $1$ . This is the smallest output possible with this function. If the input is larger than  $1$ , then the outputs are larger than  $-4$ . If the inputs are less than  $1$ , the outputs are also larger than  $-4$ .

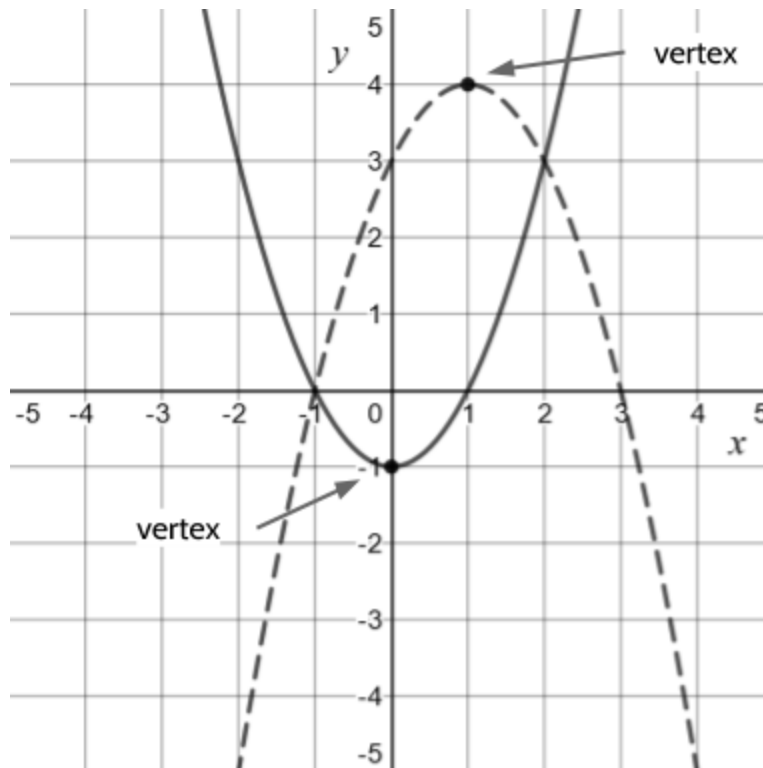
- 77) What is the smallest value possible for  $y$  in the function shown in the graph on the right?



## Tools of Algebra: Nonlinear Functions (Part 1)

With some quadratic functions, the minimum  $y$ -value marks the bottom of the parabola. Other functions have a maximum value for the outputs that marks the top of the parabola. The lowest or highest point on a parabola is called the *vertex*.

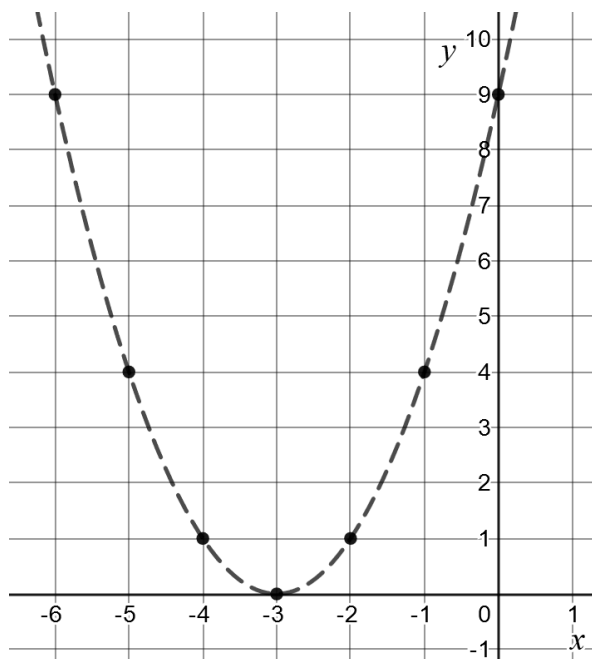
Here are two quadratic functions and their graphs. Each one has a vertex, but one is a minimum and one is a maximum. There are two *vertices* (plural of vertex) shown in the graph.



- 78) One of the functions has a maximum output value. What are the coordinates of the vertex of this function?
- A. (1, 4)                      C. (-1, 4)
- B. (4, 1)                      D. (-4, 1)
- 79) What is the minimum  $y$ -value for the other function?

Tools of Algebra: Nonlinear Functions (Part 1)

The function equation in standard form for this graph is  $f(x) = x^2 + 6x + 9$ . Notice that the graph of the function intersects with the  $y$  axis at the point  $(0, 9)$ .



- 80) Where would the function  $y = x^2 + 6x + 5$  intersect the  $y$  axis? How do you know?

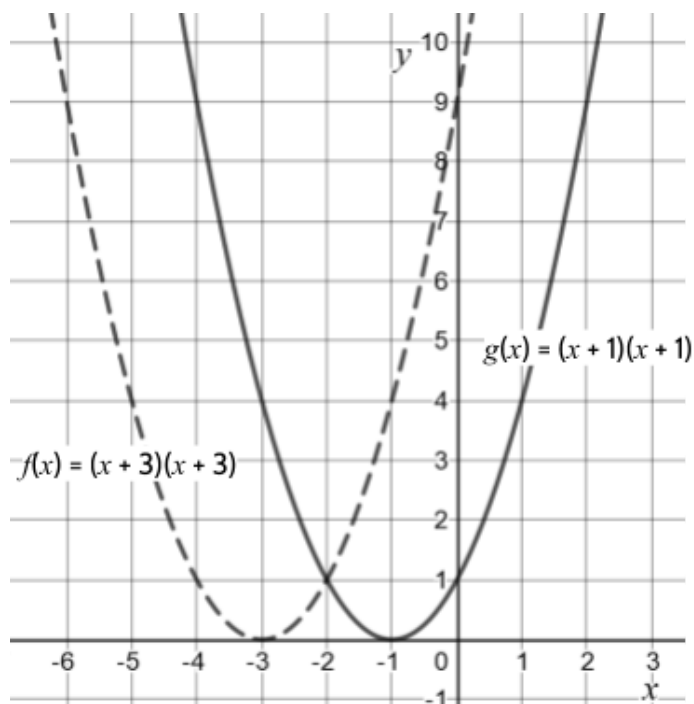
The factored form of  $f(x) = x^2 + 6x + 9$  is  $f(x) = (x + 3)(x + 3)$ . A new function  $g(x) = (x + 1)(x + 1)$  is drawn on the same graph.

- 81) What is the  $y$ -intercept of  $g(x)$ ?

- 82) How many solutions are there for  $g(x)$  and what are they?

- 83) When compared to the graph of  $f(x)$ , the graph of  $g(x)$  is

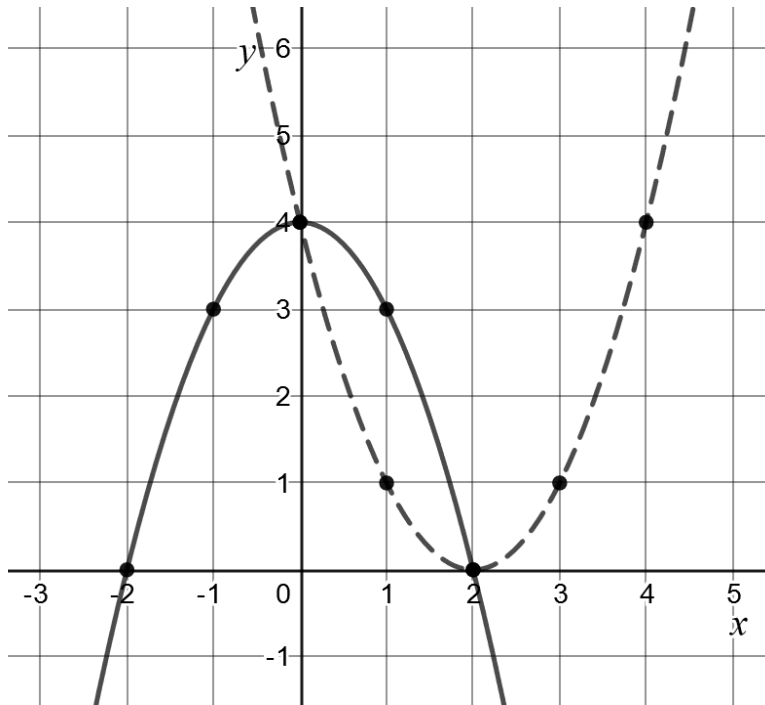
- A. shifted 2 units to the left
- B. shifted 2 units to the right
- C. shifted 8 units down
- D. shifted 8 units up



*Tools of Algebra: Nonlinear Functions (Part 1)*

Think back to everything we have done in this section on quadratic functions: tables, equations, and graphs. You might want to review the previous pages.

Then, look carefully at the graph of the two functions below and answer the question.



- 84) Tell us everything you know about the two functions shown in the graph above. Attach another piece of paper if you need room.

## Three Views of a Quadratic Function - Answer Key

1)

<i>In</i>	<i>Out</i>
1	2
2	4
3	6
4	8
5	10

2)

<i>In</i>	<i>Out</i>
1	3
2	6
3	11
4	18
5	27

3) We're interested in what difference you see between the two functions.

<i>In</i>	<i>Out</i>		<i>In</i>	<i>Out</i>
1	3	$\left. \begin{array}{l} \phantom{+} \\ \phantom{+} \\ \phantom{+} \\ \phantom{+} \\ \phantom{+} \\ \phantom{+} \end{array} \right\} +6$ $\left. \phantom{+} \phantom{+} \phantom{+} \phantom{+} \phantom{+} \phantom{+} \right\} +10$ $\left. \phantom{+} \phantom{+} \phantom{+} \phantom{+} \phantom{+} \phantom{+} \phantom{+} \right\} +14$ $\left. \phantom{+} \phantom{+} \phantom{+} \phantom{+} \phantom{+} \phantom{+} \phantom{+} \phantom{+} \right\} +18$ $\left. \phantom{+} \phantom{+} \phantom{+} \phantom{+} \phantom{+} \phantom{+} \phantom{+} \phantom{+} \phantom{+} \right\} +22$	1	2
2	9		2	5
3	19		3	9
4	33		4	14
5	51		5	20
6	73		6	27

4)

5)

<i>In</i>	1	2	3	4	5	6
<i>Out</i>	2	6	12	20	30	42

6)

<i>In</i>	1	2	3	4	5	6
<i>Out</i>	1	3	6	10	15	21

7) We are interested in the patterns you see. You can see the patterns we notice below.



*Tools of Algebra: Nonlinear Functions (Part 1)*

- 8) The rate of change is +3, +5, +7, etc. The differences between outputs are going up by 2.
- 9) The rate of change is +6, +10, +14, +18, etc. The differences between outputs are going up by 4.
- 10) The rate of change is +3, +4, +5, +6, etc. The differences between outputs are going up by 1.
- 11) The rate of change is +4, +6, +8, +10, etc. The differences between outputs are going up by 2.
- 12) The rate of change is +5 and is constant. The differences between the numbers are +5 throughout the pattern. This is a linear pattern.

13) B

14)

<i>Input</i>	<i>Input + 3</i>	<i>Input · (Input + 3)</i>	<i>Output</i>
5	5 + 3	5(5 + 3)	40
9	9 + 3	9(9 + 3)	108
20	20 + 3	20(20 + 3)	460
6	6 + 3	6(6 + 3)	54

15)

In	Calculation	Out
0	$0 \cdot 0 + 8$	8
1	$1 \cdot 1 + 8$	9
2	$2 \cdot 2 + 8$	12
3	$3 \cdot 3 + 8$	17
4	$4 \cdot 4 + 8$	24
5	$5 \cdot 5 + 8$	33

- 16) The expression  $In \cdot In + 8$  means we multiply the input by itself, then add 8. If the input is 3, our calculation would be  $3 \cdot 3 + 8 = 17$ . The expression  $In \cdot (In + 8)$  means we add 8 to the input, then multiply the result by the input. If the input is 3, our calculation would be  $3 \cdot (3 + 8) = 33$ . The results of the calculation (17 and 33) are very different. This is related to the order of operations, which you can practice in **The Tools of Algebra: Expressions, Equations, and Inequalities**.

Tools of Algebra: Nonlinear Functions (Part 1)

17)

In	Calculation	Out
0	$0 \cdot (0 + 8)$	0
1	$1 \cdot (1 + 8)$	9
2	$2 \cdot (2 + 8)$	20
3	$3 \cdot (3 + 8)$	33
4	$4 \cdot (4 + 8)$	48
5	$5 \cdot (5 + 8)$	65

18) There are different things to notice. Here are a few:

- The expressions have the same inputs, but different outputs.
- $In \cdot (In + 8)$  has smaller outputs at the beginning but grows more quickly than  $In \cdot In + 8$ .
- Both expressions have a value of 9 when the input is 1.

19) C

20) 180

21) 12

22)  $In(In + 5)$  or  $x(x + 5)$

23)

In	Calculation	Out
0	$0 \cdot (0 + 5)$	0
1	$1 \cdot (1 + 5)$	6
2	$2 \cdot (2 + 5)$	14
3	$3 \cdot (3 + 5)$	24
4	$4 \cdot (4 + 5)$	36
5	$5 \cdot (5 + 5)$	50

24) A

25) 150

26) 7

*Tools of Algebra: Nonlinear Functions (Part 1)*

27)

In	Calculation	Out
5	$5 \cdot 5 - 5$	20
8	$8 \cdot 8 - 8$	56
1	$1 \cdot 1 - 1$	0
3	$3 \cdot 3 - 3$	6
2	$2 \cdot 2 - 2$	2
10	$10 \cdot 10 - 10$	90

28)

In	Calculation	Out
3	$(3 \cdot 3) + (4 \cdot 3)$	21
5	$(5 \cdot 5) + (4 \cdot 5)$	45
2	$(2 \cdot 2) + (4 \cdot 2)$	12
8	$(8 \cdot 8) + (4 \cdot 8)$	96
7	$(7 \cdot 7) + (4 \cdot 7)$	77
4	$(4 \cdot 4) + (4 \cdot 4)$	32

29)

In	Calculation	Out
7	$(7 + 4)^2$	121
5	$(5 + 4)^2$	81
6	$(6 + 4)^2$	100
0	$(0 + 4)^2$	16
3	$(3 + 4)^2$	49
10	$(10 + 4)^2$	196

30) 256

31) 1

32) D

Tools of Algebra: Nonlinear Functions (Part 1)

33)

Rule: $y = (x + 2)(x + 3)$		
$x$	Calculation	$y$
-5	$((-5) + 2) \cdot ((-5) + 3)$	6
-4	$((-4) + 2) \cdot ((-4) + 3)$	2
-3	$((-3) + 2) \cdot ((-3) + 3)$	0
-2	$((-2) + 2) \cdot ((-2) + 3)$	0
-1	$((-1) + 2) \cdot ((-1) + 3)$	2
0	$(0 + 2) \cdot (0 + 3)$	6
1	$(1 + 2) \cdot (1 + 3)$	12
2	$(2 + 2) \cdot (2 + 3)$	20
3	$(3 + 2) \cdot (3 + 3)$	30
4	$(4 + 2) \cdot (4 + 3)$	42
5	$(5 + 2) \cdot (5 + 3)$	56

Rule: $y = x^2 + 5x + 6$		
$x$	Calculation	$y$
-5	$(-5)^2 + 5(-5) + 6$	6
-4	$(-4)^2 + 5(-4) + 6$	2
-3	$(-3)^2 + 5(-3) + 6$	0
-2	$(-2)^2 + 5(-2) + 6$	0
-1	$(-1)^2 + 5(-1) + 6$	2
0	$(0)^2 + 5(0) + 6$	6
1	$(1)^2 + 5(1) + 6$	12
2	$(2)^2 + 5(2) + 6$	20
3	$(3)^2 + 5(3) + 6$	30
4	$(4)^2 + 5(4) + 6$	42
5	$(5)^2 + 5(5) + 6$	56

There are many things to notice. One of the things you might notice is that the outputs for both tables are the same. That is because  $y = (x + 2)(x + 3)$  and  $y = x^2 + 5x + 6$  are equivalent functions.

34) B

35) A

36) 2

37) 10

38) 6

39) There are two possible  $x$ -values (-3 and -2) when  $y = 0$ .

40) The equation  $y = (x + 2)(x + 3)$  includes the numbers 2 and 3, the positive versions of -2 and -3, which are the solutions (or roots) to the function.

Tools of Algebra: Nonlinear Functions (Part 1)

41)

Rule: $y = (x + 5)(x - 3)$		
$x$	Calculation	$y$
-5	$((-5) + 5) \cdot ((-5) - 3)$ $(0) \cdot (-8)$	0
-4	$((-4) + 5) \cdot ((-4) - 3)$ $(1) \cdot (-7)$	-7
-3	$((-3) + 5) \cdot ((-3) - 3)$ $(2) \cdot (-6)$	-12
-2	$((-2) + 5) \cdot ((-2) - 3)$ $(3) \cdot (-5)$	-15
-1	$((-1) + 5) \cdot ((-1) - 3)$ $(4) \cdot (-4)$	-16
0	$(0 + 5) \cdot (0 - 3)$ $(5) \cdot (-3)$	-15
1	$(1 + 5) \cdot (1 - 3)$ $(6) \cdot (-2)$	-12
2	$(2 + 5) \cdot (2 - 3)$ $(7) \cdot (-1)$	-7
3	$(3 + 5) \cdot (3 - 3)$ $(8) \cdot (0)$	0
4	$(4 + 5) \cdot (4 - 3)$ $(9) \cdot (1)$	9
5	$(5 + 5) \cdot (5 - 3)$ $(10) \cdot (2)$	20

42) -5 and 3

43) Look at the expressions  $(x + 5)$  and  $(x - 3)$ . The numbers in the expressions are the opposite of the zeros. We can also see the zeros or roots in the table.

44) -15

45)  $y = x^2 + 2x - 15$

46)

$$5 \times 0 = 0$$

$$8 \times 0 = 0$$

$$0 \times 0 = 0$$

$$0 \times 25 = 0$$

$$0 \times \frac{1}{4} = 0$$

$$1,000 \times 0 = 0$$

47) Each equation includes the number 0. And they are all equal to 0.

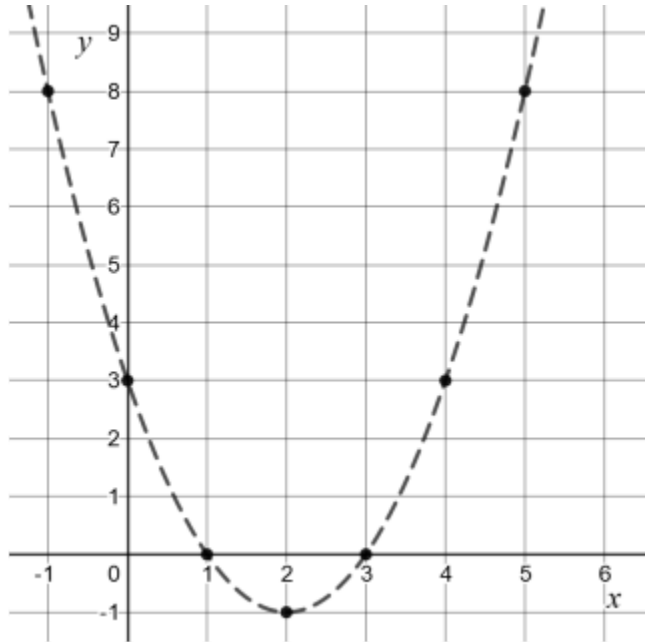
48) 2 since  $2 - 2 = 0$

*Tools of Algebra: Nonlinear Functions (Part 1)*

- 49) -7 since  $-7 + 7 = 0$
- 50) -4 and 6
- 51) -3 and 3
- 52) 2 and 4
- 53) -2 and -5
- 54) -3 (this function only has one solution)
- 55)  $\frac{1}{2}$
- 56) -6 (Be careful. The value of  $c$  is not 6. You can remember to include the subtraction as a negative sign.)
- 57) 3
- 58) 24
- 59) 15
- 60) -1 and 1
- 61) In the function  $y = 2x^2 + 6x + 4$ , if  $a$  was 0, the function would be  $y = 0x^2 + 6x + 4$ . However, 0 times  $x^2$  is 0, which means that the function would actually be  $y = 0 + 6x + 4$  or  $y = 6x + 4$ . This is a perfectly good function, but it isn't quadratic. It's a linear function. To have a quadratic function in the form  $y = ax^2 + bx + c$ ,  $a$  can be any number except for 0.

Tools of Algebra: Nonlinear Functions (Part 1)

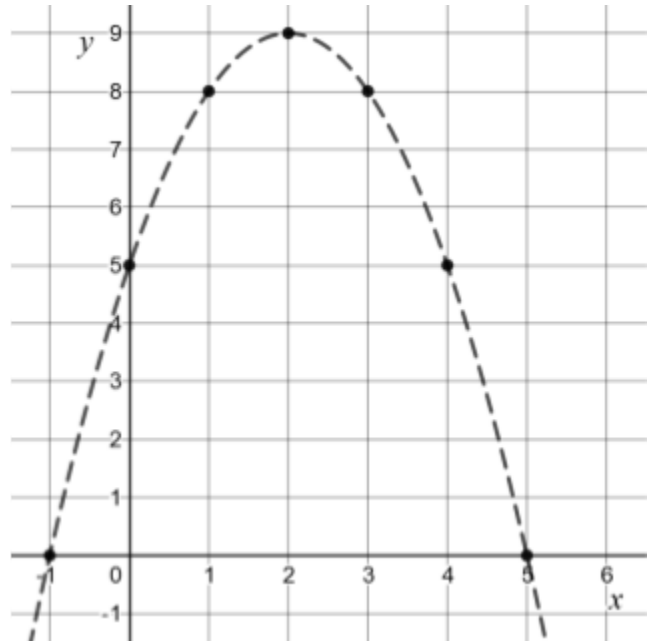
Equation: $y = x^2 - 4x + 3$		
$x$	$y$	Ordered Pair
-1	8	(-1, 8)
0	3	(0, 3)
1	0	(1, 0)
2	-1	(2, -1)
3	0	(3, 0)
4	3	(4, 3)
5	8	(5, 8)



62)

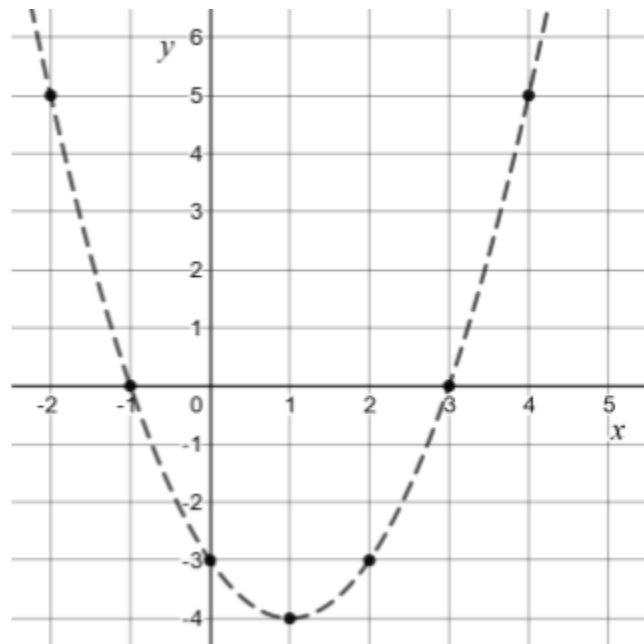
63) We want to know what you think.

Rule: $y = -x^2 + 4x + 5$		
$x$	$y$	Ordered Pair
-1	0	(-1, 0)
0	5	(0, 5)
1	8	(1, 8)
2	9	(2, 9)
3	8	(3, 8)
4	5	(4, 5)
5	0	(5, 0)



64)

Rule: $y = x^2 - 2x - 3$		
$x$	$y$	Ordered Pair
-2	5	(-2, 5)
-1	0	(-1, 0)
0	-3	(0, -3)
1	-4	(1, -4)
2	-3	(2, -3)
3	0	(3, 0)
4	5	(4, 5)



65)

66) There are many things to notice. Here are a few:

- All three graphs are curves.
- Two of the graphs have ends that point up. One of the graphs has ends that point down.
- All three graphs intersect the  $x$  axis. The  $x$ -intercepts can't be seen in this version of their equations.
- All three graphs intersect the  $y$  axis. They intersect at +3, +5, and -3, which can be seen in the equations.

67) In the first equation,  $a$  is 2. In the second equation,  $a$  is -2.

68) The first equation has ends that point up. It opens up and has a round part at the bottom. The second equation has ends that point down. It opens down and has a round part at the top.

69) D

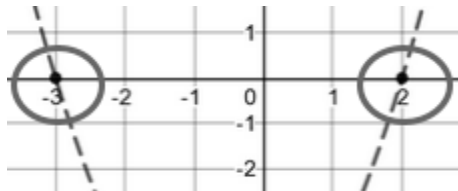


Tools of Algebra: Nonlinear Functions (Part 1)

70)

x	y
-4	6
-3	0
-2	-4
-1	-6
0	-6
1	-4
2	0
3	6

71) -3 and 2



72)

73) 1 and -1

74) The zeros should be at (-1, 0) and (3, 0).

75)

x	-2	-1	0	1	2	3	4
y	5	0	-3	-4	-3	0	5

76) You should get  $x = 3$  and  $x = -1$

77) 2

78) A

79) -1

*Tools of Algebra: Nonlinear Functions (Part 1)*

- 80) At (0, 5) or  $y = 5$ . In the standard form  $y = ax^2 + bx + c$ , the variable  $c$  is the number where the the function intersects the  $y$  axis.

You could also check this by inserting 0 for  $x$  in the function:

$$y = (0)^2 + 6(0) + 5$$

$$y = 0 + 0 + 5$$

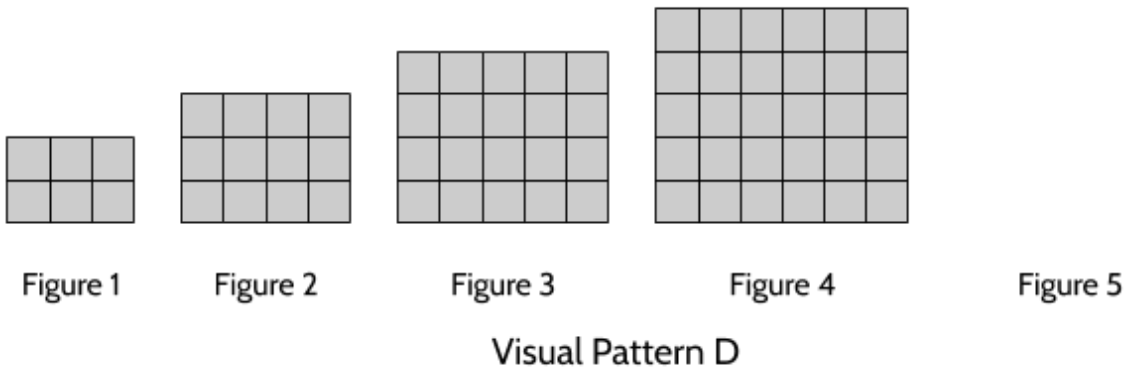
$$y = 5$$

- 81) 1
- 82) The function  $g(x)$  only has one solution: -1. The function only intersects with the  $x$  axis at one point: (-1, 0).
- 83) B. If you imagine moving  $f(x)$  to the right 2 units, it would be exactly where  $g(x)$  is.
- 84) There are many things that you can say about this graph. For example, you might be to create function tables from these graphs. You might be able to find solutions/roots/zeros. You could tell us what the vertices are of each function. You might even be able to tell us what the equations are for these functions. We look forward to reading your responses.

## Exploring Quadratic Growth

### Visual Patterns II

1) Look at the following visual pattern:



a) Draw Figure 5.

b) Describe the growth of this visual pattern.

*Tools of Algebra: Nonlinear Functions (Part 1)*

c) Complete the table.

Figure	Number of Squares
1	
2	
3	
4	
5	
6	

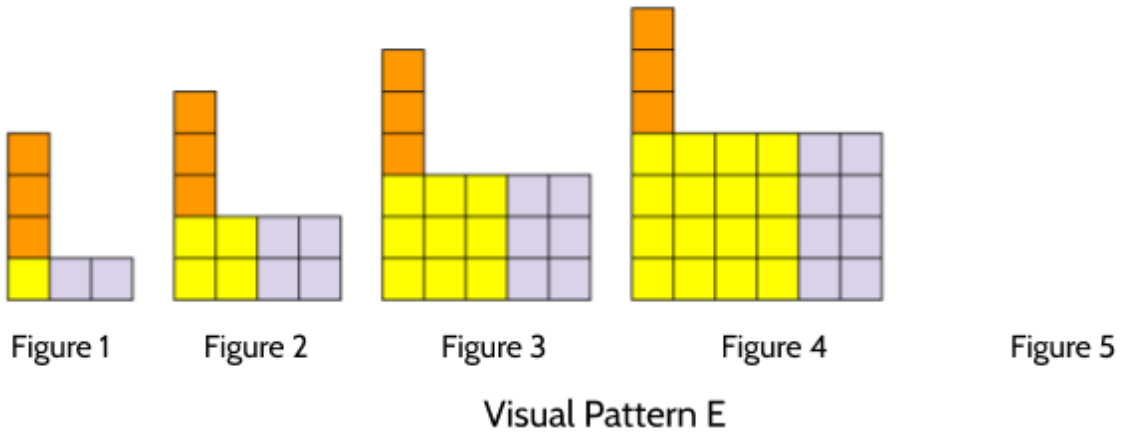
d) In a few sentences, describe what the 10th figure would look like.

e) Explain how you would figure out the number of squares in the 99th figure.

f) How many squares would be in the 43rd figure?

g) **BONUS QUESTION:** Describe how you would determine the number of squares in any figure in this pattern.

2) Look at the following visual pattern:



a) Draw Figure 5.

b) Describe the growth of this visual pattern.

*Tools of Algebra: Nonlinear Functions (Part 1)*

c) Complete the table.

Figure	Number of Squares
1	
2	
3	
4	
5	
6	

d) In a few sentences, describe what the 10th figure would look like.

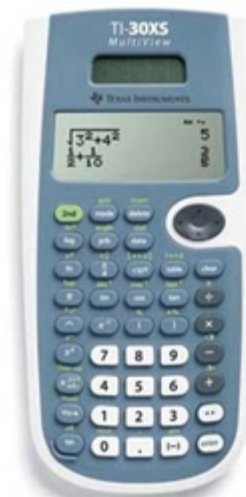
e) Explain how you would figure out the number of squares in the 99th figure.

f) How many squares would be in the 43rd figure?

g) **BONUS QUESTION:** Describe how you would determine the number of squares in any figure in this pattern.

## Squaring and Multiplying

In this activity, you will use a calculator to look for patterns when squaring and multiplying numbers. Of course, you don't have to use a calculator, but it will make it much faster if you do. We recommend the TI-30XS since it used on most high school equivalency exams.




### Raising number to the second power (AKA “squaring”)


Squaring a number is the same as raising it to the second power or multiplying the number by itself. Here are a few examples:

$$\begin{array}{lll} 5^2 = 25 & 10^2 = 100 & 15^2 = 225 \\ 5 \times 5 = 25 & 10 \times 10 = 100 & 15 \times 15 = 225 \end{array}$$

You can square numbers quickly in a calculator by using one of these two buttons:



Using the TI-30XS to find  $12^2$ , you can enter 12, then , and hit Enter. 144 should be displayed.

Another way to find 12 to the 2nd power is to enter 12, then , then 2, and hit Enter. The ^ key can be used to calculate any power (2nd power, 3rd power, 4th power, etc.)

In this exercise, you will practice squaring numbers and then make connections to algebra.

Enter a two-digit number into your calculator and then raise it the 2nd power.

For example:  $35^2 = 35 \cdot 35 = 1225$

Then, multiply the numbers that are 1 more and 1 less than the number you chose:

$$36 \cdot 34 = 1224$$

Finally, compare the square of your original number with the product of the numbers that are 1 more and 1 less.

Turn to the next page to try these steps yourself.

1 more and 1 less

3) Complete this table with different two-digit numbers.

two-digit number	the number squared	the product of 1 more than the number and 1 less than the number
35	$35 \cdot 35 = 1225$	$36 \cdot 34 = 1224$
13	$13^2 =$	$14 \cdot 12 =$

4) What do you notice?

Try to answer the following questions without a calculator, then check your answer with a calculator afterwards.

5)  $25 \cdot 25 = 625$ . What value of  makes  $26 \cdot 24 =$   true?

6)  $18^2 = 324$ . What value of  makes  $(18 + 1)(18 - 1) =$   true?

7)  $31 \cdot 29 = 899$ . What is the solution for  in  $30 \cdot 30 =$  ?

8) What value of  makes  $(9 + 1)(9 - 1) +$    $= 9^2$  true?

9) What patterns do you see when you compare  $n^2$  to  $(n - 1)(n + 1)$ ?



**2 more and 2 less**

Now try some other two-digit numbers, but this time multiply the numbers that are 2 more and 2 less than the number you choose. Then compare your answer to the number squared. For example, if you chose 35 as the number, you would multiply 37 and 33. Compare the answer with  $35^2$ .

10) Complete the table with other two-digit numbers.

two-digit number	the number squared	the product of 2 more than the number and 2 less than the number
35	$35^2 = 1225$	$37 \cdot 33 =$

11) What do you notice?

Try to answer the following questions without a calculator, then check your answer with a calculator afterwards.

12)  $25 \cdot 25 = 625$ . What value of  $w$  makes  $27 \cdot 23 = w$  true?

13)  $18^2 = 324$ . What value of  $x$  makes  $(18 + 2)(18 - 2) = x$  true?

14)  $16 \cdot 12 = 192$ . What is the solution for  $y$  in  $14 \cdot 14 = y$ ?

15) What value of  $z$  makes  $(9 + 2)(9 - 2) + z = 9^2$  true?

16) What patterns do you see when you compare  $n^2$  to  $(n - 2)(n + 2)$ ?

**3 more and 3 less**

Now try some other two-digit numbers, but this time multiply the numbers that are 3 more and 3 less than the number you choose. Then compare your answer to the number squared.

17) Complete the table with other two-digit numbers.

two-digit number	the number squared	the product of 3 more than the number and 3 less than the number
35	$35^2 = 1225$	$38 \cdot 32 =$

18) What do you notice?

Try to answer the following questions without a calculator, then check your answer with a calculator afterwards.

19)  $25 \cdot 25 = 625$ . What value of  $a$  makes  $28 \cdot 22 = a$  true?

20)  $18^2 = 324$ . What value of  $b$  makes  $(18 + 3)(18 - 3) = b$  true?

21)  $48 \cdot 42 = 2016$ . What is the solution for  $c$  in  $45 \cdot 45 = c$ ?

22) What value of  $d$  makes  $(14 + 3)(14 - 3) + d = 14^2$  true?

23) What patterns do you see when you compare  $n^2$  to  $(n + 3)(n - 3)$ ?

*Tools of Algebra: Nonlinear Functions (Part 1)*

- 24) Complete the table. Try to use patterns you found earlier.

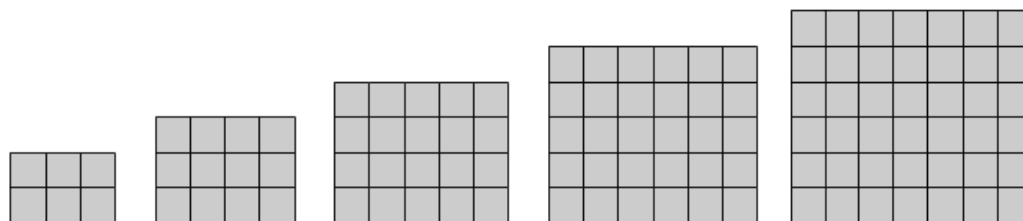
$n$	$n^2$	$(n + 1)(n - 1)$	$(n + 2)(n - 2)$	$(n + 3)(n - 3)$
5	25	24		
8				55
12			140	
16		255		
20	400			
26				667

- 25) Challenge question: Can you explain why the value of  $(n + 1)(n - 1)$  is always 1 less than  $n^2$ ? And why  $(n + 2)(n - 2)$  is always 4 less than  $n^2$ ? Etc.

In the second part of this packet, we used functions to understand quadratic growth in the world. We will come back to the example of gravity and learn how acceleration follows quadratic patterns. We will also look at other examples of quadratic growth in science, business, and everyday life.

## Exploring Quadratic Growth - Answer Key

## 1) Visual Pattern D



a) Figure 1      Figure 2      Figure 3      Figure 4      Figure 5

b) There are many ways to describe the growth. Here are a few:

- The number of squares in each figure is increasing by 6, then 8, then 10, then 12...
- A column and a row are added in each figure.
- Figure 1 is  $2 \times 3$ , Figure 2 is  $3 \times 4$ , Figure 3 is  $4 \times 5$ , etc. The height and width grow by 1 each time.
- The height is 1 more than the figure number. The width is 2 more than the figure number.

c) Complete the table.

Figure	Number of Squares
1	6
2	12
3	20
4	30
5	42
6	56

- d) The 10 figure would be 11 rows in height and 12 columns wide. In other words, it would be a rectangle with a height of 11 units and a width of 12 units. Your way of describing the 10th figure could be different.
- e) There are different ways to figure out the number of squares in the 99th figure.

## Tools of Algebra: Nonlinear Functions (Part 1)

- You could create a table with inputs and outputs all the way to Figure 99. It would take a while, but it would work.
- You could draw a rectangle with a height of 100 and a width of 101. The area of this rectangle is the number of squares.
- You might have another way that works just as well.

f) The 43rd figure would have 1,980 squares:  $44 \cdot 45 = 1980$

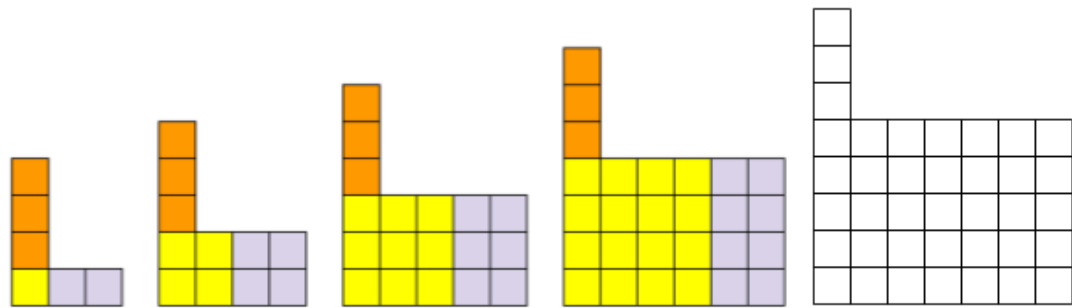
g) You could write a function using the figure number as an input and the number of squares as an output. Here are a few versions:

$$\text{Number of Squares} = (\text{Figure number} + 1)(\text{Figure number} + 2)$$

$$y = (x + 1)(x + 2)$$

$$f(x) = (x + 1)(x + 2)$$

### 2) Visual Pattern E



a) Figure 1      Figure 2      Figure 3      Figure 4      Figure 5

b) There are many ways to describe the growth. Here are a few:

- The number of squares in each figure is increasing by 5, then 7, then 9, then 11...
- The vertical 3 squares stay the same in each figure.
- There is a square in the bottom left that is growing in each figure:  $1 \times 1$ ,  $2 \times 2$ ,  $3 \times 3$ ,  $4 \times 4$ , etc.
- A column and a row are added to the base of each figure. The base is growing like this:  $1 \times 3$ ,  $2 \times 4$ ,  $3 \times 5$ ,  $4 \times 6$ ,  $5 \times 7$ , etc.
- The height grows by 1 in each figure: 4, 5, 6, 7, 8, etc.
- The width grows by 1 in each figure: 3, 4, 5, 6, 7, etc.

*Tools of Algebra: Nonlinear Functions (Part 1)*

- The height is 3 more than the figure number. The width is 2 more than the figure number.

c) Complete the table.

Figure	Number of Squares
1	6
2	11
3	18
4	27
5	38
6	51

- d) The 10th figure would have a base rectangle that is 10 units high and 12 units wide. On the top left, there would be 3 squares sticking up vertically. Your way of describing the 10th figure could be different.
- e) There are different ways to figure out the number of squares in the 99th figure.
- You could create a table with inputs and outputs all the way to Figure 99. It would take a while, but it would work.
  - You could draw a base rectangle with a height of 99 and a width of 101. The area of this rectangle is the number of squares in the base. Then you could add the 3 squares on top.
  - You might have another way that works just as well.
- f) The 43rd figure would have 1,938 squares:  $43 \cdot 45 + 3 = 1938$
- g) You could write a function using the figure number as an input and the number of squares as an output. Here are a few versions:

$$\text{Number of Squares} = (\text{Figure number})(\text{Figure number} + 2) + 3$$

$$y = (x)(x + 2) + 3$$

$$f(x) = x^2 + 2x + 3$$

3)

two-digit number	the number squared	the product of 1 more than the number and 1 less than the number
35	$35 \cdot 35 = 1225$	$36 \cdot 34 = 1224$
13	$13^2 = 169$	$14 \cdot 12 = 168$

Tools of Algebra: Nonlinear Functions (Part 1)

4) *The product of 1 more than the number and 1 less than the number is always 1 less than the number squared.*

5) 624

6) 323

7) 900

8) 1

9) The value of  $n^2$  is always 1 more than the value of  $(n - 1)(n + 1)$ .

10)

two-digit number	the number squared	the product of 2 more than the number and 2 less than the number
35	$35 \cdot 35 = 1225$	$37 \cdot 33 = 1221$

11) *The product of 2 more than the number and 2 less than the number is always 4 less than the number squared.*

12) 621

13) 320

14) 196

15) 4

16) The value of  $n^2$  is always 4 more than the value of  $(n - 2)(n + 2)$ .

17)

two-digit number	the number squared	the product of 3 more than the number and 3 less than the number
35	$35 \cdot 35 = 1225$	$38 \cdot 32 = 1216$

18) *The product of 3 more than the number and 3 less than the number is always 9 less than the number squared.*

19) 616

20) 315

*Tools of Algebra: Nonlinear Functions (Part 1)*

21) 2025

22) 9

23) The value of  $n^2$  is always 9 more than the value of  $(n - 3)(n + 3)$ .

$n$	$n^2$	$(n + 1)(n - 1)$	$(n + 2)(n - 2)$	$(n + 3)(n - 3)$
5	25	24	21	16
8	64	63	60	55
12	144	143	140	135
16	256	255	252	247
20	400	399	396	391
26	676	675	672	667

24)

25) Hint: The expressions  $(n + 1)(n - 1)$ ,  $(n + 2)(n - 2)$ , and  $(n + 3)(n - 3)$  are in factored form. Try changing them to standard form. What do you notice?



## Vocabulary Review

### Glossary

You can use this section to look up words used in this math packet.

**consecutive** (adjective): Numbers which follow each other one after the other, without gaps. The numbers 3, 4, 5, 6 are in consecutive order.

**constant** (noun): A value in an expression or equation that doesn't change. For example, in the function  $y = 4x + 6$  the numbers 4 and 6 are both constants.

**constant** (adjective): Something that stays the same. A *constant* rate of growth means that the rate stays the same over time.

**distributive property of multiplication** (noun): a mathematical property that shows multiplying a number by a group of numbers added together is the same as doing each multiplication separately. For example,  $3 \cdot (2 + 4)$  is equal to  $3 \cdot 2 + 3 \cdot 4$ .

**equal sign**: a symbol used to show symmetric balance between two values or quantities, one on each side of the equal sign. Can be read as "is equivalent to" or "is the same as".

**equation** (noun): A number sentence that shows two expressions are equal by using the equal sign.  $2^3 = 8$  is an *equation*.  $5x + 3$  is an expression, not an *equation*.

**equivalent** (adjective): Having the same value. For example, 4 quarters and 20 nickels are equivalent. Eight hours is equivalent to 28,800 seconds.  $4^3$  and 64 are equivalent.

**estimate** (verb): To make a rough guess at a number, usually without making written calculations

**evaluate** (verb): To calculate the value of something. If asked to *evaluate*  $4^3$ , your answer should be 64.

**expression** (noun): Numbers and symbols that show the value of something. 100,  $5x + 3$ , and  $2^3$  are all *expressions*.  $5x + 3 = 23$  is an equation made up of two *expressions*.

## Tools of Algebra: Nonlinear Functions (Part 1)

**exponent** (noun): In a quantity represented as a power, the *exponent* shows how many times the base is multiplied. The exponent is shown as a smaller number up and to the right of the base. For example, in the power  $2^3$ , the *exponent* is 3.

**exponential growth** (adjective): A growth pattern related to multiplying by a repeated factor.  $y = 2^x$  is a common example of an exponential function where 2 is used as a factor  $x$  times to calculate the output of  $y$ .

**factor** (noun): Whole numbers that are multiplied together to get another number. A number that can be divided into another number evenly, with no remainder.

**factor** (verb): To split a number into its factors (see above definition of factors).

**function** (noun): a relationship between two quantities. For every input into a function, there can only be one output. A function can be represented by an equation, a table, a graph, and a verbal or written description.

**generalize** (verb): To look at specific examples and realize that something is true *in general*. For example, what happens with you divide a number by itself? Specific examples:  $8 \div 8 = 1$  or  $25 \div 25 = 1$  or  $.75 \div .75 = 1$ . Example of a *generalization*: Any number divided by itself is 1.

**graph** (noun): A visual representation of a function relationship.

**input** (noun): The number that goes into a function.

**linear function** (noun): A function that has a constant rate of change. The graph of a linear function is a straight line.

**multiple** (noun): A number that can be divided by another number evenly, with no remainder. 25 is a *multiple* of 5.

**nonlinear function** (noun): A function that does not have a constant rate of change. The graph of a nonlinear function is not a straight line. There are many kinds of nonlinear functions. Two of them are *quadratic* functions and *exponential* functions.

**ordered pair** (noun): Ordered pairs are used to show a position on a graph. An ordered pair is made up of an input and an output that make a function true. They are usually written in parentheses like this (14, 7), where the first number shows the  $x$  value (horizontal) and the second number shows the  $y$  value (vertical).

**output** (noun): The value that comes out of a function when each input goes in. The output is dependent on the input.

**parabola** (noun): The shape of the graph of a quadratic function.

**pattern** (noun): Patterns are things we can observe. We gather information about what has come before to make predictions about what will come next.

**property** (noun): A character or quality that something has. In science, we use the physical property to refer to color, texture, density, and other qualities of physical objects. In mathematics, properties refer to characteristics of numbers or operations. Example:

*Commutative property:* Addition and multiplication are *commutative*, meaning the order of the operation doesn't matter.  $2 + 7 = 7 + 2$  and  $2 \cdot 7 = 7 \cdot 2$

**product** (noun): The result of multiplication. 4 times 5 gives a *product* of 20.

**quadratic growth** (noun): A growth pattern related to the size of different squares. A quadratic function in standard form looks like  $y = ax^2 + bx + c$  where  $a$  can be any number except 0. The  $x^2$  ( $x$  "squared") term in the equation makes the function quadratic.

**rate of change** (noun): The rate of change in the outputs when the inputs are consecutive. Known as *slope* when a part of the graph of an equation.

**root** (noun): A *root* is a value you can put into a function as an input for  $x$  so that 0 is an output for  $y$ . For example, the  $x$ -values -3 and -2 are the roots of the function  $y = (x + 2)(x - 3)$ . In a graph, the  $x$ -intercepts (points where a line intersects with the  $x$  axis) are the same as the *roots*. The word *zero* is sometimes used to mean the same thing as *root*. The roots,  $x$ -intercepts, and zeros (all which mean the same thing) are the solutions to a function equation.

*square root:* A square root of a number is a value that, when multiplied by itself, gives the number. The *square root* of 25 is 5.

**starting amount** (noun): The value of the output when the input is zero. Known as *y-intercept* when a part of the graph of an equation.

**term** (noun): A single number or variable, or numbers and variables multiplied together. Terms are separated by + or - signs. For example, there are three terms in the equation  $2 + 5 = 7$ .

## *Tools of Algebra: Nonlinear Functions (Part 1)*

**variable** (noun): A letter or symbol that represents another value, either any number, a specific number, or a set of numbers. In the expression  $x^3$ ,  $x$  is a *variable* that could mean any number. Variables can also represent other things. For example, in geometry, points and angles are represented by letters.

**vertex** (noun): The highest or lowest point on a parabola.

**x-axis**: The line that goes from left to right in a graph. The horizontal axis on the coordinate grid.

**x-intercept** (noun): The point of intersection with the  $x$ -axis of a graph.

**y-axis** (noun): The line that goes from top to bottom in a graph. The vertical axis on the coordinate grid.

**y-intercept** (noun): The point of intersection with the  $y$ -axis of a graph.

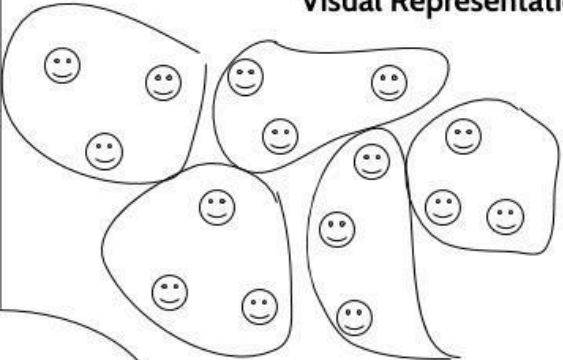
**zero-product property** (noun): This fact has to do with multiplication. If two numbers multiplied together gives a result of 0, then one of the two numbers must equal zero. It is also written this way: If  $a \cdot b = 0$ , then  $a$  or  $b$  has to be equal to 0.

## Using Graphic Organizers to Learn Vocabulary

In order to learn math vocabulary, we need practice using words in different ways. In this activity, you will choose a few words from this packet that you want to practice, then you will complete a graphic organizer for each word. Look at the sample for the word *quotient* below.

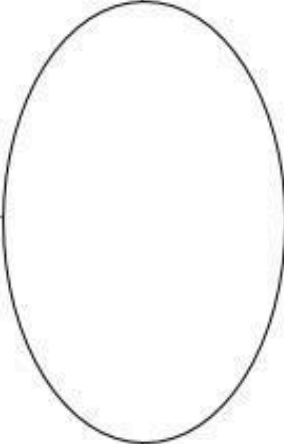
To start, choose a word from the packet and complete the graphic organizer:

- **What is the definition of the word?** You can look at the vocabulary review on page 111 for help. Write the definition in your own words to really make the word yours.
- **Make a visual representation.** You can make a drawing or diagram that will help you remember what the word means.
- **What are some examples of the word you're studying?** Below you can see that there are examples of *quotients*, which are the answers to division problems.
- **What are some non-examples of this word?** These are things that are **not** the word you're studying. For example, 24 is **not** the quotient of 4 divided by 6.

<p><b>What is it?</b></p> <p>A quotient is the result of dividing one number by another. It is the answer to a division question.</p>	<p><b>Visual Representation</b></p> 
<p><b>What are some examples?</b></p> <p>15 divided by 3 equals 5</p> <p><math>66 \div 6 = 11</math></p> <p><math>63/18 = 3.5</math></p> <p>5, 11 and 3.5 are quotients in these calculations.</p> <p>population <math>\div</math> area = <b>density</b></p>	<p><b>What are some non-examples?</b></p> <p>4 times 6 equals 24</p> <p><math>18 + 5 = 23</math></p> <p><math>17 - 2.5 = 14.5</math></p> <p><math>3.5 \times 18 = 63</math></p>

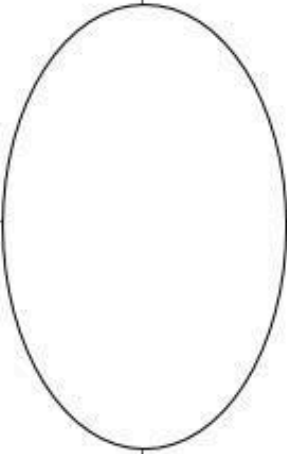
*Tools of Algebra: Nonlinear Functions (Part 1)*

<p>Visual Representation</p>	<p>What are some non-examples?</p>
<p>What is it?</p>	<p>What are some examples?</p>



*Tools of Algebra: Nonlinear Functions (Part 1)*

<p>Visual Representation</p>	<p>What are some non-examples?</p>
<p>What is it?</p>	<p>What are some examples?</p>



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## **Versions**

V.1.3, 9/14/2021:

- Moved answer keys to the end of each section for consistency with other packets

V.1.4, 5/3/2022:

- Formatting corrections

V.1.5, 10/13/2023:

- Added Assessment and Placement Recommendations
- Revised Welcome! letter for consistency with other packets
- Moved vocabulary graphic organizer to the Vocabulary Review section for consistency with other packets