UNIT 7

Equality

Rationale

Equality is a fundamental concept in algebra. It is noted through use of an equal sign, represents a relationship of equivalence, and can be conceptualized by the idea of balance. Despite the importance of the equal sign, there is tons of research that shows that students often have serious misconceptions about what the equal sign means. The research has mostly been conducted with elementary through high school students, but using an assessment included in this unit, I have found it is equally true of adult learners.

Many adult education students think the equal sign means "the answer" or "what you get when you do the operation." Students who think this way tend to answer questions like **12 + 5 = ☐ + 6** incorrectly, writing a 17 in the square. Students also get thrown when they see things like **14 = 17 – ☐**, because the operation is taking place on the right side of the equation.

This misunderstanding makes sense if you think about students' prior experiences with the equal sign. They have been seeing equality in arithmetic for years; most of it has looked like **7 × 8 =**___ or **25 × 18 =** ____. Even the use of a calculator can reinforce this misconception—just push the "=" and the answer appears.

This association of the equal sign with "the answer" as opposed to understanding it as signifying balance or equivalence can be especially problematic when students encounter the symbol in algebra. If students understand the equal sign to mean balance, it makes sense to perform the same operations on either side of the equation. Without an understanding of a balance point between the two sides of the equation, the rule, "Whatever you do to one side you have to do to the other," becomes a disconnected rule that students struggle to memorize and use.

The problems in this unit were chosen to help students understand the meaning of the equal sign. This is not something we can just tell students and expect to fully take root—they have a deeply-seated misconception that won't be corrected easily. These activities are intended to help draw out students' intuitive sense for keeping things in balance and connecting that to the formal use and notation of the equal sign.

Students don't need to know the names or the symbolic representations, but a sense of these properties of equality will help them gain flexibility in understanding balance and equivalence:

- the reflexive property of equality *($a = a$)*
- the transitive property of equality (If a = b and $b = c$, then $a = c$)
- the substitution property of equality (if $a = b$, then b can replace a in any expression without changing the value of that expression)
- the addition property of equality (If $a = b$, then $a + c = b + c$)
- the subtraction property of equality (If $a = b$, then $a - c = b - c$)
- the multiplicative property of equality (If $a = b$, then $a \times c = b \times c$)
- the division property of equality (If $a = b$, then $a \div c = b \div c$)

OPPORTUNITIES TO DEVELOP THE FOLLOWING SKILLS

- **Developing fluency with the different properties of equality.**
- **Creating equations with two or more variables to represent relationships between quantities.**

KEY VOCABULARY

balance: a state of equilibrium; equal distribution of weight, amount, etc. Have students brainstorm situations when we talk about "balance."

a balance scale: a device for weighing. When the pans are equivalent in weight, the scale will be level. We can replace objects in the pans with objects of equal weights without changing the balance. We place an object in one pan and standard weights in the other to find what the object weighs.

equivalent: having the same value. For example, 4 quarters and 20 nickels are equivalent. Eight hours is equivalent to 28,800 seconds.

equal sign: a symbol used to show symmetric balance between two values or quantities, one on each side of the equal sign. Can be read as "is equivalent to" or "is the same as".

Core Problem Overview: Zip Zap Zowie

This problem describes the relationship (in terms of weight) between four kinds of imaginary objects: zips, zaps, zowies, and swooshes. We are told that one zip equals three zaps. Two zaps equals five zowies. Three zowies equals two swooshes. Finally, we are told that a swoosh weighs 60 pounds. This problem is a good way to draw out “working backwards” as a problem-solving strategy. This is the most common path students find to a solution, though it may take many students some time and trial and error before they come to that strategy.

In the end, students should be able to explain how we can tell that each zip weighs 300 pounds, each zap weighs 100 pounds, and each zowie weighs 40 pounds.

A word on the words: It’s fine to change the words if you think your students might be thrown by the imaginary words. I like the “zip,” “zap,” “zowie” and “swoosh” for the lightheartedness of students having to say things like, *“Well, I know two swooshes weigh 120 pounds and that is equivalent to 3 zowies, but how do I figure out how much a zowie weighs?”*

TEACHING THE CORE PROBLEM

Have a few students read the problem aloud while others follow along. Give them time to work on their own—maybe five minutes. Then have them get into groups of three or four, share their thinking so far, and then keep working towards a solution. Students should be given time to struggle with this problem, and you should support any method that the student has chosen.

If students are struggling, suggest that they try to create a visual representation of the situation. They might draw shapes or pictures, or create some other visual model. A great way to support students with this problem is to give them materials to make the situation tactile as well as visual. Offer students stickies (of different colors if possible), index cards, pattern blocks, etc. to let them experiment with different visualizations. To the {direction} you can see an example of a visual model of a group’s solution method.

In addition to suggesting students try to make visual representations, you can help students get unstuck by asking 1 or 2 of the following clarifying questions:

- *What do we know?*
- *What does it mean to say “one zip weighs the same as three zaps?”*
- *If one swoosh weighs 60 pounds, how much do two swooshes weigh?*
- *How much do three zowies weigh? How do you know?*
- *How many zaps weigh the same as 2 swooshes and 2 zowies?*

- *Which weighs more, a ____ or a ____? How do you know?* (You can ask students to compare any of the four words).

Students will need some sense of the properties of equality to work on this problem. This is not to say that you should present these properties before you give out this problem. Many of your students either know, or will intuit, the properties they need. Others will need more support, and we should ask questions to help them see useful properties wherever they are in their solution process. As groups are working, walk around and look for opportunities to test understanding. For example, asking *If one swoosh weighs 60 pounds, how much do two swooshes weigh?* or *How much do three zowies weigh? How do you know?* will help students think about the reflexive property of equality. The idea that each swoosh (and zowie, and zap, and zip) weighs the same amount as every other swoosh is important and may not be immediately understood by every student.

You may have a group that finishes the problem early while others are still working. Ask questions of different group members to make sure they can all explain the method. If each member can, you can offer the following extension, which I learned from Patricia Helmuth, an adult education math teacher in New York's Hudson Valley. Ask students to create another situation (with real-life objects or imaginary words) using the same relationships between the elements, but where $d \neq 60$.

$$a = 3b$$
$$2b = 5c$$
$$3c = 2d$$

Note: Students will probably not solve this problem by setting up equations. That is something that we will draw out of the debrief on their work. Given that, it's best not to use equations in the phrasing of the extension.

PROCESSING THE PROBLEM

As you are walking around and asking the groups clarifying questions, be on the lookout for two things: (1) moments where groups get stuck and (2) different strategies students use.

When you ask groups to present their work, start with the most visual and concrete methods. Save the group that used an equal sign in their method for last. You probably won't get a group that sets up algebraic equations with variables, but you might have something like:

zip = 3 zaps
2 zaps = 5 zowies
3 zowies = 2 swooshes

Ask each group to prepare to present their reasoning by putting their method on chart paper/newsprint. Ask them to make their method clear so that someone else could understand it without having to ask them any questions. As the groups present their strategies, give the class time to ask questions and make statements of appreciation about each other's' methods. If no group uses the equal sign, introduce the notation after all the groups have presented. Either way, ask the rest of the class what they think about the use of the equal sign.

Recommended Extension: To engage them further, you can ask students to use the information in the problem and use the equal sign to show equivalence in other ways.

For example:

1 zap = 1 swoosh and 1 zowie
1 zip = 5 swooshes

How many zowies and swooshes would it take to balance with a zip?

Supplemental Problems

The supplemental problems in this unit continue the work of the core problem, giving students the opportunity to build on the strategies and discoveries they've made.

- **Assessment on Equality**
  This assessment can give you insight into your students' understanding of the equal sign. I am still always surprised by how many of these students get wrong. It really goes a long way towards understanding the problems they have working with equations.

- **Balancing the Scales**
  These four problems allow students to explore the concept of equality using balance scales. They come from *Math Matters* (see resource section). Each problem has three scales. Students consider the information in the first two scales to figure out what it would take to balance the third scale. Students will have the opportunity to use the addition/subtraction/ multiplication / division and substitution properties of equality.

- **Noah's Ark**
  This problem comes from fawnnguyen.com, a great math teaching blog by Fawn Nguyen, a middle school teacher in California. It is similar to the core problem, and the balancing scales problems but it is more complex, with more objects (animals) and more "equations." I highly recommend offering students tools to make visual representations of the problem. You might even make extra sheets of animals for students to cut out. This is a great problem and a great way to see how students can put together all the strategies they developed working on the other problems in this unit.

- **Fix These Equations**
  I really like doing this activity, created by Steve Hinds, director of Active Learning in Adult Numeracy (alanproject.org). It helps students practice the order of operations, while strengthening their sense of equations being equivalent. In general, having students evaluate whether equations are true or false is a good way to focus their thinking and see where they are at in their understanding of the equal sign. This activity presents a series of false equations—that is, none of them are true. The goal is for students to experiment with placement of parentheses to make the equations true. This is a nice way to end an exploration of the equal sign, since this problem goes beyond the sense of equivalence and actually uses the formal notation of the equal sign.

Zip Zap Zowie

1 zip weighs as much as 3 zaps.

2 zaps weigh as much as 5 zowies.

3 zowies weigh as much as 2 swooshes.

If one swoosh weighs 60 pounds, how many pounds does a zip weigh?

Equality Assessment

Write the correct answer on the line.

9 + 11 = _____ + 6

_____ = 17 − 4

19 + _____ = 21 + 4

14 − 4 = _____ − 3

45 + 13 = 13 + _____

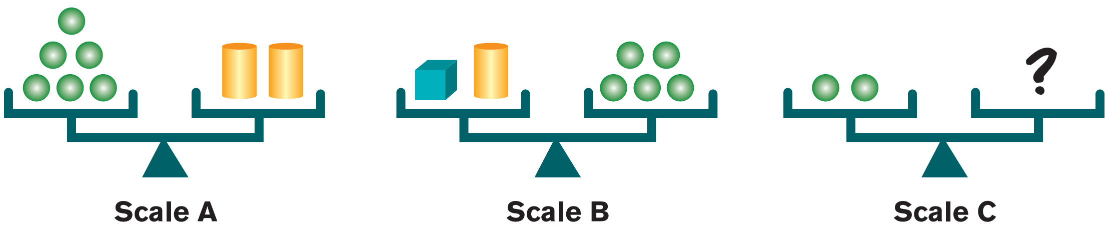
_____ + 10 = 7 + 9

Describe what this symbol = means without using the word "equal."

Keeping Your Balance

Solve the four balance problems below. In each problem, use the information from the balanced scales A and B to figure out what is needed to balance scale C.

1



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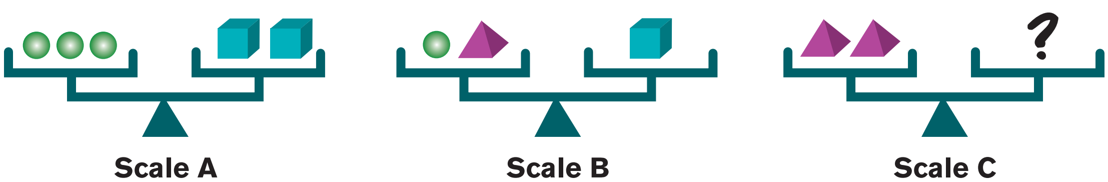

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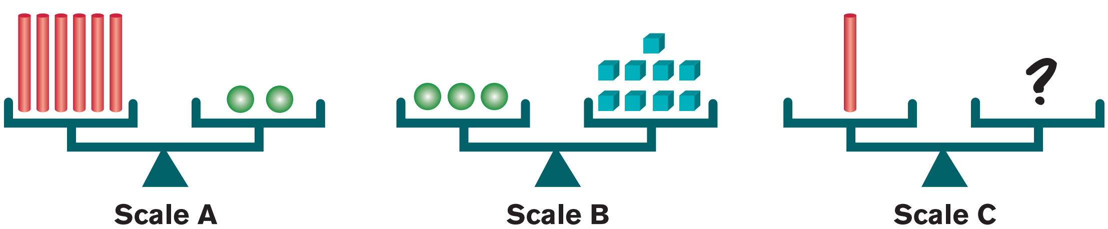

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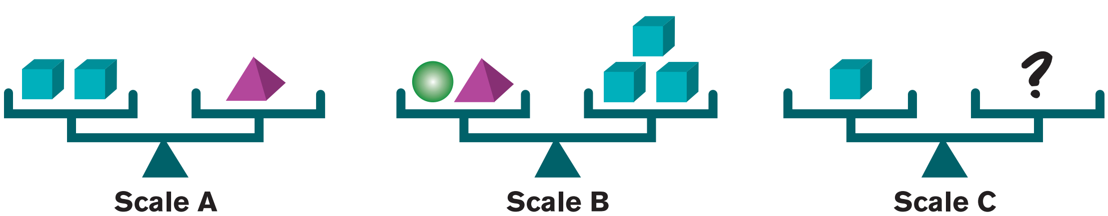
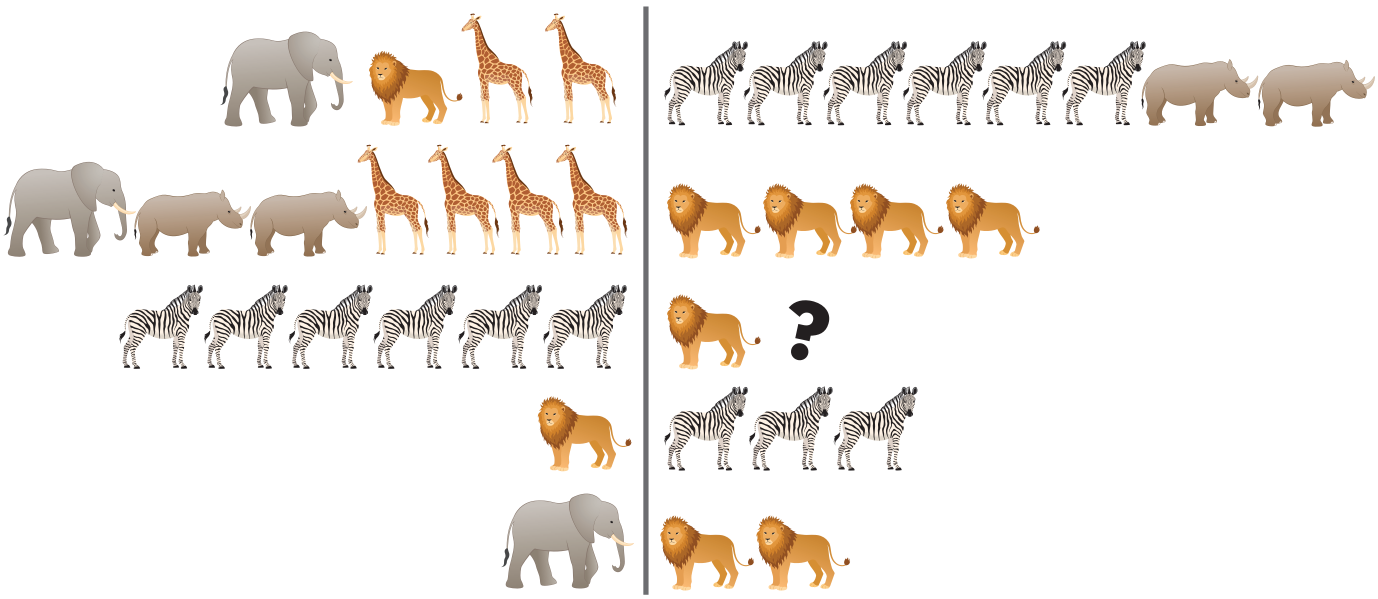


Adapted from *Math Matters: Understanding the Math You Teach* by Suzanne Chapin and Art Johnson

Noah's Ark

Mr. Noah wants his Ark to sail along on an even keel. The ark is divided down the middle, and on each deck the animals on the left exactly balance those on the right—all but the third deck.

Can you figure out how many giraffes are needed in place of the question mark so that they (and the lion) will exactly balance the six zebras?



Adapted from *fawnnguyen.com*

Fix These Equations!

None of the following equations is correct. Insert parentheses so that they are correct.

a. $13 - 2 \times 5 = 55$

b. $12 = 3 \times 6 - 2$

c. $11 - 2 \times 4 + 1 = 1$

d. $11 - 3 \times 4 + 2 = 34$

e. $23 = 3 + 7 \times 2 + 3$

f. $12 - 2 \times 5 + 1 = 60$

g. $4 - 1^2 - 5 = 4$

h. $8 + 2 \times 4 - 1 = 14$

i. $12 - 8 \times 1 + 7 = 32$

j. $8 - 2 + 6 \div 3 = 4$

k. $7 + 3^2 = 100$

l. $24 + 16 \div 8 - 4 = 10$

m. $20 \div 7 - 2 + 5^2 \times 3 = 79$

Solutions to Fix These Equations!

Traditional work with students on the order of operations can be very dull. In this format, students must use the order of operations constantly, but it has the advantage of a puzzle-like quality.

a. $(13 - 2) \times 5 = 55$

b. $12 = 3 \times (6 - 2)$

c. $11 - 2 \times (4 + 1) = 1$ *(This one is difficult.)*

d. $(11 - 3) \times 4 + 2 = 34$

e. $23 = (3 + 7) \times 2 + 3$

f. $(12 - 2) \times (5 + 1) = 60$

g. $(4 - 1)^2 - 5 = 4$ *(If your students haven't worked with exponents, call this one a bonus problem.)*

h. $8 + 2 \times (4 - 1) = 14$

i. $(12 - 8) \times (1 + 7) = 32$

j. $(8 - 2 + 6) \div 3 = 4$

k. $(7 + 3)^2 = 100$ *(See letter g)*

l. $(24 + 16) \div (8 - 4) = 10$

m. $20 \div (7 - 2) + 5^2 \times 3 = 79$