UNIT 5 — Nonlinear Functions

Up until now, the problems in this curriculum framework have focused on linear relationships and functions, and students have explored these relationships through tables, rules, and graphs. In this unit, we introduce the concept of nonlinear functions. The core problem in this unit, the *Job Offer Problem*, will help students to see how the graph of a quadratic or polynomial function differs from the graph of a linear function, and they will develop a more nuanced understanding of rate of change. Students will also work with quadratic functions that model real-world applications to physics.

SKILLS DEVELOPED

- Using charts, tables, drawings, and graphs to analyze nonlinear change.
- Graphing quadratic functions and relating them to function tables.
- Generating outputs from a rule written in function notation.
- Understanding constraints on the domain (possible inputs) of a function given a particular context.
- Seeing and recognizing the difference between the graphs of quadratic and cubic functions.
- Applying quadratic function rules in specific problem-solving contexts.

KEY VOCABULARY

nonlinear function: a function in which the variable is raised to the power of 2 or higher. The graph of a nonlinear function forms a curve.

parabola: the shape formed by the graph of a quadratic function. Parabolas are U-shaped and can open either upward or downward.

polynomial function: a mathematical expression with two or more terms. Often, polynomial functions are written as the sum or difference of several terms in which the same variable is raised to different powers.

quadratic function: a polynomial function in which the variable is raised to the power of 2, but not higher. The graph of a quadratic function forms a parabola.

cubic function: a polynomial function in which the variable is raised to the power of 3.

Core Problem Overview:
The Job Offer Problem

The *Job Offer Problem* asks students to use math to help them make a decision. They have to choose between two job offers based on each company's plan for calculating raises. Using math to make and justify financial decisions is an important life skill—one that is especially emphasized in Unit 4, as well as other units in this curriculum framework.

The problem begins by asking students to think about both offers, without doing much, if any, math. They should consider the offers as they are presented and write about why they would choose one job offer over the other. At this stage in the problem, it is important that students understand exactly how the raise plan works at Big Time Publishing. At BTPC, Rachel's hourly wage will be $0.50 more than it was the year before. This means that after the first year, she will get a raise of $0.50 an hour. After two years, she will receive a raise of $1.00 an hour, and after three years she will get a $1.50/per hour raise. The pattern continues, with the amount of Rachel's raise increasing after each additional year she works for the company. The raise schedule at Nadir Books is linear—that is, Rachel will always receive a raise of $1.50 per hour for each year that she works for the company. At this stage in the activity, your role is to help students understand how the different raises work. You don't need to help them calculate her hourly wage in a given year. Just get them thinking about how the raises work so that they can make a decision about which job they would choose.

The problem includes scaffolding in the form of a table to help students answer question number 2. Depending on the level of your class, you may choose to remove this scaffold and ask them to solve the problem in any way they see fit. One benefit of using the table is that it helps students to draw the graphs for Rachel's salary at Big Time versus Nadir Books, which is the next step in the problem. The main goal of the *Job Offer Problem* is to help students see what nonlinear growth looks like when graphed, and so this is a key step in the problem. Once students have completed the table and finished drawing their graphs,
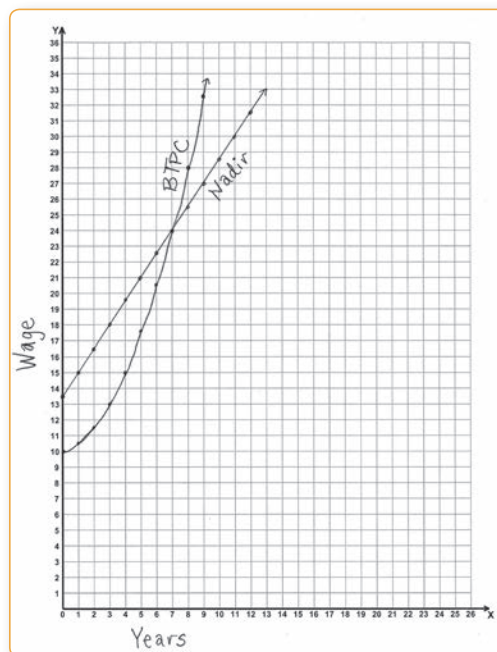
it would be a good idea to spend time talking about the similarities and differences in the graphs. For example, both represent an increase, but the rate of increase for Big Time Publishing Company changes each year, which results in a curve.

The last part of the problem asks students if they would still choose the same offer as they did in the first part. Here, students should have a conversation about why they would stick with their choice or why they might change it. For example, a student who had initially chosen Nadir Books as the better offer might switch to Big Time Publishing, pointing out that it would pay off eventually if she worked at Big Time for several years. Others might point out that it is risky to plan ten or fifteen years ahead when taking a job because of the economy or changes in consumer demand. All of this is rich territory for you to explore with your class, and it helps students see how math can be used in decision-making.

TEACHING THE CORE PROBLEM

After students have had time to read the first part of the problem, you should help them to clarify exactly how the raises work at both companies. You might begin by asking some questions like:

- *Let's start by talking about the offer at Big Time. Can someone explain to me how the raises work at this company?*

- *How big of a raise does Rachel get after one year at Big Time? After two years? After four years?*

- *How much of a raise would Rachel get at Nadir Books after one year? After two years? After four years?*

Once students understand how the raises work, ask them to take a few minutes and decide which job they would want to take. You could then ask students to discuss their choice with a partner, or you could ask the class to vote and record their responses on the board.

Next, you should give the students time to work individually on the second part of the problem. While they are working, make sure to circulate around the room and observe each student's work. Their table for Big Time Publishing should reflect the whole-class discussion that took place earlier. If you find that a student is having trouble calculating Rachel's correct wage for a given year, you can ask questions like:

- *Talk to me about the wages for Big Time in your table. How did you calculate those?*

- *Let's read the prompt together. For each year at Big Time, Rachel's raise will be $0.50 more than it was the year before. What does that mean?*

■ *Let's think about the discussion we had earlier. We said
that Rachel raise would get bigger every year at Big Time.
Does your table show this?*

After students complete the table for Big Time, they may have trouble
seeing that the raise for Nadir is the same year after year. If you notice
students making mistakes on this part of the table, you can use a
similar line of questioning or ask them about the rate of change.

You'll notice that the axes on the graph paper are not labeled, and you
may need to help your students identify which axis should be used
to represent years (the *x*-axis) and which should be used to represent
Rachel's wage (the *y*-axis). It might help them to ask which column
in the table represents the inputs (the years) and which represents
the outputs (the hourly wages) in this situation. If students are really
struggling as a result of the two columns of outputs, you can have them
re-write the table as two different tables—one for BTPC and one for
Nadir Books. Ultimately, your students should recognize that the graphs
should be increasing as they move from left to right, and this might help
them to realize how to label the axes. If they label the axes incorrectly
and start to draw graphs that are decreasing, you can talk to students
about making sure that what's happening in the graph reflects what's
happening in the table—that is, if wages are going up on the table, the
graphs should be going up as well.

When students finish the graphing part of the activity, have them talk
about the differences in the graphs. Some questions you might ask them
to consider are:

■ *What is similar about these two graphs? What is different?*
■ *What do you notice about how the graph of Rachel's wages
at Big Time changes over time?*
■ *Does this make you want to change your vote from the first part
of the problem? Why?*
■ *What do you think would happen to the graphs after the ten years
shown on the table? Can you extend them on the graph paper?*

If some students finish early, you could give them calculators and
ask them to solve an extension question. You could use the following
questions:

■ *Let's say Rachel would be paid for 35 hours per week, for 50 weeks
a year. How much would she make at each company in the first
year after she was hired?*
■ Some students might even notice that, while she would make
more during her seventh year at Big Time, she also would be

making less each year for the six years before that. In this case, you could ask, *Rachel knows that she wants to choose a job and stick with it for ten years. How much in total earnings would she make after ten years at Big Time? How much would she make at Nadir?*

■ You might also ask students to think about how long Rachel would have to work at Big Time to make her choice really pay off: *If Rachel worked 35 hours a week for 50 weeks a year, how many years before her total earnings at BTPC were more than her total earnings at Nadir Books?*

These extensions are by no means exhaustive, but they help students to think further about the idea of choosing a job.

PROCESSING THE PROBLEM

Because the goal of this problem is for students to see the difference in a graph that grows linearly and one that grows quadratically, there is not a lot of room for multiple solution strategies. Your main goal in talking about this problem with the class will be to address the shapes of the two graphs. But first, it would be a good idea to discuss question number 4 as a group. You might start by asking how many people changed their mind about which offer they would accept. Some questions to get the discussion started are:

■ *How many people changed their vote? And if you changed it, what made you decide to change? If you don't want to change your vote, why not?*

■ *Do we all agree on which company gave Rachel the better offer? Why do you think we disagree?*

After each student has had a chance to contribute to the discussion, tell the group that you want to talk about the shapes of the graphs. You should let the students talk about all the things that the graph have in common and all of the ways that they are different. You might write a dividing line through the middle of the whiteboard and list the similarities on the left and the differences on the right. The main similarity is that both graphs represent an increasing wage, but there are a number of differences that could come up in the discussion. For example, one of the graphs is a curve while the other is a line; the two graphs start at different points on the y-axis; and the graph of Big Time increases slowly at first but then gets steeper and steeper, while the graph of Nadir increases at the same rate.

Now you can ask some specific questions about the graphs. Some suggestions for the whole-class discussion are:

- *What were some of the things you noticed from the graph?*

- *During the first few years, which graph increases the most rapidly?*

- *During which year do the graphs increase at the same rate?*

- *When does the graph of Big Time start to grow more rapidly than the graph of Nadir?*

- *What will happen if we continue the graph for Nadir Books? What will happen if we continue the graph for Big Time? How do you know?*

- *How does the graph help support your decision of which offer to take?*

When you feel like your students have a good understanding of the difference between the linear and quadratic graph, talk to your students about some of the vocabulary you will be using to talk about graphs like this one. You could introduce the terms *nonlinear* and *parabola* and explain that students will be seeing them again in the supplemental problems and on the HSE exam.

SUPPLEMENTAL PROBLEMS

The supplemental problems in this unit invite further investigation into the graphs of nonlinear functions. I would recommend doing these problems in the order that they are presented here, though each of these should be accessible to students who have worked on the *Job Offer Problem.*

- **The Nebraska Rainfall Problem**
  This problem asks students to take information from a table, put it onto a graph, and draw conclusions about rainfall levels based on the shape of the graph. Initially, the graph increases at a quadratic rate. It then levels off, which indicates that no significant rainfall occurred during that period. Then at the end of the year, the graph increases at a linear rate. Students will plot the points carefully and answer questions about the changes in rainfall from month to month. The problem also asks them to tell the story of how hard the rain fell throughout the year. In teaching this problem, you would want them to see that the rain fell at an increasingly heavy rate in the first half of the year, then there was no rainfall, and then the rain fell at a constant rate at the end of the year. We have included a graph that shows Yearly Precipitation alongside Monthly Precipitation. You might distribute these to your students after they have completed the activity so that you can talk about the relationship between the graphs.

■ **Gravity and a Dropped Ball**

This supplemental problem introduces algebraic notation to illustrate that the velocity of a dropped object will increase the longer it falls. The goal of this problem is to build some background knowledge in physics and help them get familiar with evaluating quadratic functions on the TI-30XS Multiview calculator. Students will begin by completing a chart that shows the dropped ball's position after a given number of seconds. They will see the ball falls faster and faster over time. The last question here is tricky: By plugging 10 and 11 seconds into the formula, they will see that the ball hits the ground before 11 seconds elapse, but they may need some guidance about how to solve for the exact value. When using this problem initially, it might be good enough to help students approximate the time it would take for the ball to hit the ground. Some students may be able to come to an approximate answer by recognizing and continuing patterns in the table.

■ **The Graphs of $y = x^2$ and $y = x^3$**

In order to be successful on the HSE exam, your students will need to be able to distinguish between different kinds of nonlinear graphs. This simple activity builds upon the function machines and graphing activities from the previous units by asking students to complete an input/output table and then draw a simple quadratic and cubic graph. If you've had the chance to work with your students on signed numbers, they should be able to complete the tables without the aid of a calculator. But this can also be a good place to help students develop fluency with the TI-30XS Multiview calculator. After your students have completed each table and drawn the graphs, it would be worthwhile to talk to them about how the shapes of the graphs are similar and how they are different. You can also ask students what patterns they see in the tables (for one, there is no constant rate of change in the outputs).

Core Problem

The Job Offer Problem

Rachel is unhappy at her job, and so she has started interviewing with other employers in her industry, hoping that she can start to make more money in the long run. When she met with Big Time Publishing Company (BTPC), the manager told her that her starting wage will be $10.00 per hour and that she will receive a $0.50 raise after the first year. For each year that she stays with the company after that, her raise will be $0.50 more than it was the year before.

After her interview with Big Time Publishing Company, she met with Nadir Books and told the manager about the offer she was given. The manager assured Rachel that her offer will be better. She offered Rachel a starting wage of $13.50 per hour, with a raise of $1.50 each year.

1 Consider both offers. If you were in Rachel's shoes, which offer would you accept? Explain the reasons for your choice.

2 Complete the table below for each job offer. In which year will Rachel make the same hourly wage no matter which job she chooses?

Year	Hourly Wage at Big Time Publishing	Hourly Wage at Nadir Books
0 (newly hired)		
1		
2		
3		
4		
5		
6		
7		
8		
9		
10		

3 On graph paper, plot the points for the graph of Big Time Publishing Company's salary offer, and then connect them. Next, plot the points of Nadir Books's offer on the same graph and connect them. What do you notice about the two graphs?

4 What are the coordinates of the point where the two lines intersect? What does this point represent in the context of the problem?

5 Would you still choose the same job offer as you did before? Explain the what circumstances might make you choose the job at Big Time Publishing. Explain what circumstances might make you choose Nadir Books.

Supplemental Problem 1

The Nebraska Rainfall Problem

Rainfall is very important to farmers in the Western United States since crops like wheat, corn, or soybeans need rain water to survive. A farming family in Western Nebraska recorded the amount of rainfall on their land over a period of one year beginning in January. They kept track of the amount of rainfall that fell each month (Monthly Precipitation) in order to find out how much rain had fallen so far that year (Yearly Precipitation).

The family needs help calculating the Yearly Precipitation. For each month, calculate how much rain had fallen so far that year.

| Month	Monthly Precipitation (in inches)	Yearly Precipitation (in inches)
January	0.5	
February	1.0	
March	1.5	
April	2.0	
May	2.5	
June	3.0	
July	0.0	
August	0.0	
September	1.0	
October	1.0	
November	1.0	
December	1.0	

Using the *x-axis* for months (1 for January, 2 for February, and so on) and the *y-axis* for total rainfall, construct a graph that shows Yearly Precipitation. You should plot 12 points and connect them.

1 In which months did the total amount of rainfall grow at a linear rate? How can you tell?

2 In which months did the total amount of rainfall grow at a quadratic/nonlinear rate? How can you tell?
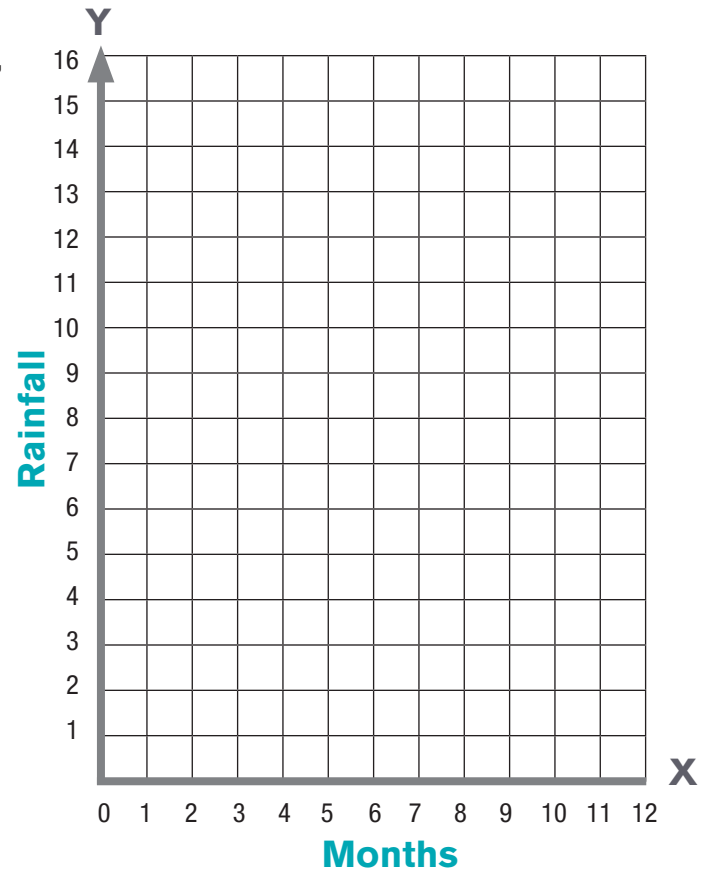
3 Use the table and your graph to tell the story about the rate at which the rain fell in Western Nebraska during this year. As specifically as you can, you should discuss the changes in rainfall—and in the shape of the graph—over the course of the year.
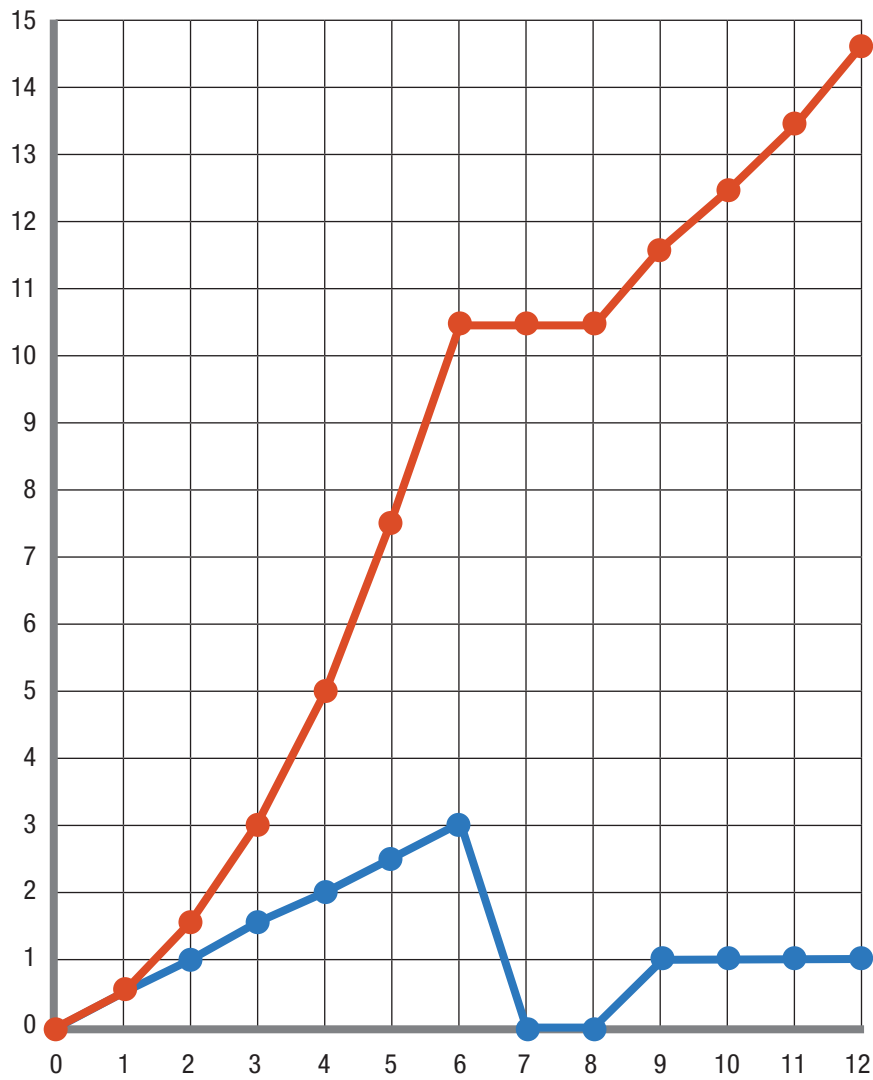
Supplemental Problem 2

Gravity and a Dropped Ball

When an object is dropped, gravity pulls it down toward the earth. To calculate the distance travelled by a falling object, we use the equation:

$$d = \tfrac{1}{2}gt^2$$

In this equation, the variable *d* represents the distance travelled by the object, *g* represents the acceleration due to gravity in meters per second squared, and *t* represents the time in seconds. On earth, the acceleration due to gravity is 9.81 meters per second squared. This never changes.

You go to the top of One World Trade Center, which is 546 meters tall, and drop a baseball from the top.

1 How long do you think it would take to hit the ground? Write your best guess.

2 Using the function equation above, complete the table below to find how far the ball will have fallen after a certain amount of time has elapsed.

Time (in seconds) *t*	Distance the ball has fallen (in meters) *d*
1	
2	
3	
4	
5	
6	
7	
8	

3 What do you notice about the velocity of the ball as it falls?

4 Does the ball fall at the same velocity the entire time? How do you know?

5 About how long will it take for the ball dropped off the top of One World Trade Center to hit the ground?

Supplemental Problem 3
The Graphs of x^2 and x^3

Complete the input/output tables below, then graph each function on the coordinate plane.

$f(x) = x^2$	
Input x	Output $f(x)$
−4	
−3	
−2	
−1	
0	
1	
2	
3	
4	
	25

$f(x) = x^3$	
Input x	Output $f(x)$
−3	
−2	
−1	
0	
1	
2	
3	