SAN DIEGO CITY SCHOOLS

Instruction & Curriculum
Mathematics

MIDDLE LEVEL

Mathematics Routine Bank

Distributed September 2004

MIDDLE LEVEL ROUTINES IN MATHEMATICS: NUMBER SENSE AND ALGEBRAIC THINKING

What is a routine?

A routine is a whole-class structured activity that gives students the opportunity to develop over time any or all of the following:

- Sense of the relative size and value of very large and very small numbers
- Operational sense
- Fluency (efficiency, accuracy, and flexibility)
- Good intuition about numbers and their relationships
- Reasoning
- Problem solving
- Mental math

Routines are a time to:

- Preview new concepts
- Review concepts that have been previously explored
- Practice those concepts which continue to be fragile

The mathematical focus of the routine may be independent of the daily launch, explore and summarize or it may set the stage for the next learning experience.

What time of the instructional day? How long does a routine last?

- Routines last approximately 8-10 minutes per day.
- A math lesson could begin or conclude with a routine.
- In a block schedule, a routine might occur midway through the time period.

How many different routine activities should be done during the year?

- It is advisable to focus in depth on several rich routines that are repeated throughout the year.
- Most of the routines should be introduced early in the year so students can become comfortable with the format of each routine.
- As students become comfortable with the format, learning will become more efficient and the mathematics rather than the format will be the focus.

Why do routines?

The Mathematics Framework for California Public Schools (2000) lists six major goals for students to achieve in mathematics. These goals are as follows:

1. Develop fluency in basic computational and procedural skills, an understanding of mathematical concepts, and the ability to use mathematical reasoning to solve mathematical problems, including recognizing and solving routine problems readily and finding ways to reach a solution or goal when no routine path is apparent.

2. Communicate precisely about quantities, logical relationships, and unknown values through the uses of signs, symbols, models, graphs, and mathematical terms.

3. Develop logical thinking in order to analyze evidence and build arguments to support or refute hypotheses.

4. Make connections among mathematical ideas and between mathematics and other disciplines.

5. Apply mathematics to everyday life and develop an interest in pursuing advanced studies in mathematics and in a wide array of mathematically related career choices.

6. Develop an appreciation for the beauty and power of mathematics.

These goals can feel like a tall order when we consider the vast number and rigor of standards that students are expected to experience and the pacing schedule that must be adhered to if students are to have experiences with all-important standards prior to the California Standards Test.

Routines provide valuable time for practice that develops fluency.

What is fluency?

Fluency includes three ideas: efficiency, accuracy, and flexibility.

- *Efficiency* implies that the student does not get bogged down in too many steps or lose track of the logic of the strategy. An efficient strategy is one that the student can carry out easily. What is efficient for one student may not be efficient for another student. What is efficient for the teacher may not be efficient for all students.

- *Accuracy* depends on several aspects of the problem-solving process, among them careful recording, knowledge of number facts and relationships and double-checking results.

- *Flexibility* requires knowledge of more than one way to solve a particular kind of problem. Students need to be flexible in order to choose an appropriate strategy for the problem at hand. Students also need to be able to use one method to solve a problem and another method to double-check the results.

Fluency rests on a well-built mathematical foundation with three parts:

- An understanding of the meaning of the operations and their relationships to each other.
- The knowledge of a large repertoire of number relationships.
- A thorough understanding of the base-ten number system, how numbers are structured in this system, and how the place value system of numbers behaves in different operations.

On-going routines allow students the opportunity to develop mathematical fluency.

What is quality practice?

- Practice refers to a variety of rich problem-based tasks or experiences, spread over numerous class periods, each addressing the same basic idea.
- Practice helps students become comfortable and flexible with an idea.
- Practice emphasizes fluency.
- Practice is NOT mindless drill of computational procedures that rely on rote memory with little understanding. Lengthy drills of algorithmic skills tend to diminish flexibility and reflective thought.

What are the outcomes of quality practice?

- An opportunity to become fluent and automatic in the use of an effective problem solving strategy.
- An increased opportunity to develop conceptual ideas and more elaborate and useful connections.
- An opportunity to develop alternative and flexible strategies.
- A greater chance for all students to understand, not just a few.
- A clear message that mathematics is about figuring this out and making sense.
- More opportunities to develop flexible, efficient, accurate skills.

Routines are an excellent opportunity to build on-going rich practice into the instructional program.

How can routines and practice assist with standardized tests?

- On standardized tests, mental computation, estimation, or a nonstandard approach is often twice as fast as memorized algorithms, especially when the format is multiple-choice.
- The distracter choices on standardized tests are designed to match the typical errors that students make in mindless application of the algorithms.
- Students who have practiced thinking are much less likely to make these errors and will actually be able to complete these tests more quickly and with greater accuracy.

Routines allow students to gain confidence, practice thinking and improve accuracy.

<u>Resources</u>:

Mathematics Framework for California Public Schools, 2000.
Russell, Susan Jo, "Developing Fluency," from Russell, Susan Jo, *Relearning to Teach Arithmetic: Addition and Subtraction, A Teacher's Study Guide,* Palo Alto: Dale Seymour Publications, 1999.
Van de Walle, John, *Elementary and Middle School Mathematics, Teaching Developmentally*, Boston, Pearson Education, 2004.
Van de Walle, John, *Elementary and Middle School Mathematics, Teaching Developmentally*, Boston, Pearson Education, 2001.

MIDDLE LEVEL

ROUTINES IN MATHEMATICS:
NUMBER SENSE AND ALGEBRAIC THINKING

Standards	Routines									
	Number of the Day	Number Lines	Number Strings	Number Talks	Concepts of Equality	Thinking Relationally	Math Trains	How Far? How Do You Know?	Close, Far, In Between	Number Translations
Whole Numbers*	●	●	●	●	●	●	●	●	●	●
Rational Numbers**	●	●	●	●	●	●	●	●	●	●
Integers***	●	●	●	●	●	●	●	●	●	
Variables				●	●	●				
Place Value	●	●	●	●		●	●	●	●	
Comparison and Ordering		●	●		●			●		●
Estimation and Rounding		●								
Operations/Calculations + - x ÷ Exponents	●		●	●	●	●	●		●	●
Functions					●	●				
Expressions/Equations	●		●		●	●				

* Whole numbers are 0, 1, 2, 3, 4, 5, 6 and so on. If a number has a decimal part, a part that is a fraction, or a negative sign, it is not a whole number.

** A rational number is any number that can be expressed as a fraction.
A rational number is any number that can be written as either a terminating or repeating decimal number. Fractions, decimals, and percents are all rational numbers.

*** Integers are the set of whole numbers and their opposites. . . . –5, -4, -3, -2, -1, 0, 1, 2, 3, 4, 5 . . .

Number Routines

• Number of the Day

• Number Lines

• Number Strings

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Number of the Day

Purpose:

- Number composition and part-whole relationships
  (for example, 10 can be 2 x 5, 4 + 6, or 20 – 10)
- Equivalent arithmetical expressions
- Different operations
- Seeing number relations, ways of deriving new numerical expressions by systematically modifying prior ones (for example, 7 + 8 = 15 so 17 + 8 = 25)

Description:

Give the students a number to consider each day. Students think of different ways to make that number. As you develop different number concepts in class, encourage your students to incorporate these concepts into the different ways they develop the number. Many students will be familiar with *Number of the Day* from past experience in mathematics classrooms. This provides an opportunity to build on that understanding and for the teacher to build in constraints as noted on the following page.

Materials:

- Chart paper
- Individual white boards or journals

Directions:

Getting started: Choose a number for the day (e.g., the number of days the children have been in school). Ask students to tell you everything they know about that number. For example: 24; the number of sodas in 4 six-packs; the number of eggs in 2 dozen, the number of crayons in my box of crayons, the number of classes at my school. Initially you will want to use *Number of the Day* in a whole group. After a short time, students will be familiar with the routine and be ready to use it independently.

1. Post the chart paper.
2. Write the *Number of the Day* at the top of the chart paper.
3. Ask students to think of several models and equations that would represent the *Number of the Day.*
4. Ask students to represent the *Number of the Day* in at least four different ways.
5. The students will document these in their daily math journals.

6. The teacher will observe the students' work and purposefully choose students to share out those representations that will move the class toward a better understanding of number and operational sense.
7. The teacher will strategically call on those students who represented the number in meaningful ways and write on the chart paper what they dictated. The teacher then leads a class conversation around those representations that best connect to concepts recently learned.

<u>Example</u>: The Number of the Day is 12

6 + 6 = 12
12 = 22 – 10
3 x 4 = 12
2^2 x 3 = 12
1/2 x 24 = 12
4.35 + 7.65 = 12
-18 + 30 = 12

5 + 5 + 2 = 12
12 = 10 + 10 – 8
100 – 80 - 8 =12
36 ÷ 3 = 12
144 ÷ 12 = 12
2 x 2 x 3 = 12
53 + (-41) = 12

<u>Constraints</u>

When students are familiar with the structure of *Number of the Day*, connect it to the number work they are doing in particular units. Add constraints to put on the sentences to practice and reinforce different mathematical concepts. Ask students to include:

- Multiplication and division
- Both addition and subtraction
- Three numbers
- Combinations of 10
- Doubles
- Doubles plus one
- Multiples of 5 and 10
- Zero property
- Emphasize using tens
- Emphasize using hundreds
- Order Property
- Commutative Property
- Associative Property
- Distributive Property
- Write equations with answer and equal sign on the left (10 = 3 + 3 + 4)
- Absolute value
- Square numbers/ Square root
- Factorials
- Show the number in different arrays
- Prime factorization of a number
- Exponents in equations
- Larger numbers, up to millions in equations.
- Ask students to construct number lines containing the number.

- Include decimals in equations.
- Include fractions in equations.
- Use friendly percents of the number (50 %, 25% 10%, 1%)
- Showing the number in different arrays.
- Using “real world” examples (e.g., dozen, week, pints in a quart, minutes in 1/2 hour, lbs. in a ton.

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Number Lines

Purpose:

- To understand relationships between numbers
- To understand the relative magnitude of numbers.

Description:

Students place numbers on a number line. Students use what they know about one number to determine where a second number should be placed. As the types of numbers change and as the scale changes, students must use reasoning skills and their understanding of amounts and quantities to place the numbers.

Materials:

- A large, blank number line easily visible to all students during the routine time
- Attached blackline master of number lines

Time: 10 minutes

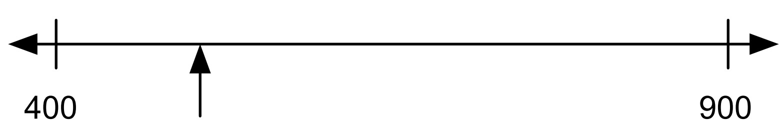
Caution: Always include arrows on both ends of your number line representations so students realize the number line is infinite; we are only looking at a section of the number line.

Directions:

Variation 1: Estimation

1. Label 2 marks on the number line (e.g., 400 and 900).

2. Place an arrow somewhere between the 2 marks.



3. The class suggests reasonable values for the number at the arrow. The students should give reasons why the numbers they suggest are reasonable (e.g., “It looks like the arrow is about one-fourth of the distance between 400 and 900. Since there are

500 numbers between 400 and 900, the arrow looks like it might be pointing to a number about 125 larger than 400, so I think it might be 525.”).

Scaffold for Variation 1:

Give the students several numbers from which to choose. Students select the number that makes the most sense to them and explain their reasoning. For example:

The arrow is pointing to which of the following numbers? Support your response with a mathematically convincing argument.

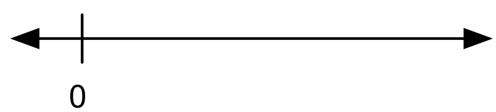
850, 490, 380

Guiding questions for Variation 1:

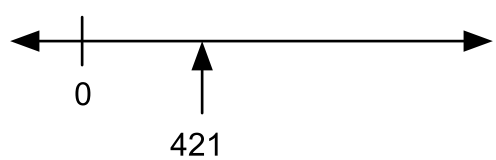
- Support your placement with a mathematically convincing argument.
- Name a number that is greater than this number.
- How much greater? Prove it on the number line.
- Name a number that is less than this number.
- How much less? Prove it on the number line.

__Variation 2__**: Estimation**

1. Label the mark on the left with a zero.



2. Tell the students the arrow is pointing to a particular number (e.g., The arrow is pointing to 421).



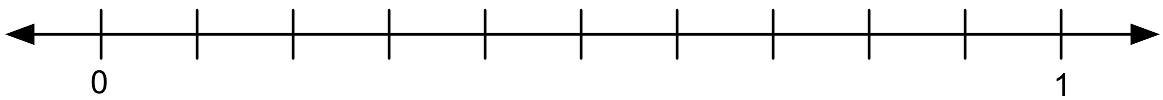
3. Ask where other numbers would be. This helps students look at the relative positions of values. For example:

   About where would 835 be?
   About where would 212 be?
   About where would 315 be?

Justify your answers with mathematically convincing arguments.

<u>Variation 3:</u> Decimals

1. Draw a number line with 11 marks, evenly spaced (this will give you ten intervals). Label the extremes "0" and "1."



2. Write decimals on index cards (one decimal number per card). Use decimals such as 0.46, 0.523, 0.7, 0.444, 0.8, 0.48, 0.32, 0.6, 0.08, etc.

3. Choose only 2 or 3 decimals to work with each time you do this routine. Make multiple copies of the same numbers. Give a card to each pair of students. Give students time to discuss where their number would make sense on the number line.

4. Have one pair of students place their card where they think their number belongs on the number line. Students must give a mathematically convincing argument as to why they are placing the number at this location.

5. Students discuss with their partners whether they agree or disagree with the placement of the card and why.

6. Class asks clarifying questions to the pair in the front of the room.

7. Students share other strategies.

8. Leave the numbers on the number line from one day to the next so that students can look at the decimals relative to other decimals with which they have worked.

Scaffold for Variation 3:

- Use decimal numbers that go to the same decimal place (e.g., all tenths, hundredths, or thousandths).

Extensions for Variation 3:

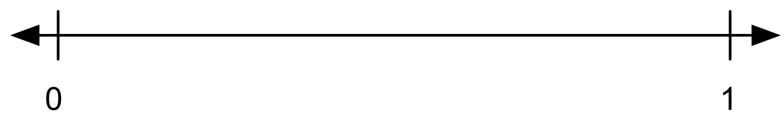
- Use labels other than 0 and 1 for the extremes (e.g., 3.1 and 4.1; 0.42 and 0.43).
- Include decimals that end in different places (i.e., 0.3, 0.36, 0.367)
- Use a number line with the intervals unmarked.

Guiding questions for Variation 3:

- Support your placement with a mathematically convincing argument.
- Name a decimal less/greater than yours.
- How do you know your decimal is less/greater than one half?
- Name another decimal equivalent to yours.

<u>Variation 4:</u> Fractions

1. Label the extremes with “0” and “1.”



2. Write different fractions on index cards (e.g., 1/8, 1/4, 3/8, 1/2, 5/8, 3/4, 7/8, 8/8 as well as their equivalent fraction names).

3. Choose only 2 or 3 fractions to work with each time you do this routine. Make multiple copies of the same numbers. Give a card to each pair of students. Give them time to discuss where their number would make sense on the number line.

4. Have one pair of students place their card where they think their number belongs on the number line. Students must give a mathematically convincing argument as to why they are placing the number at this location.

5. Students discuss with their partners whether they agree or disagree with the placement of the card and why.

6. Class asks clarifying questions to the pair in the front of the room.

7. Students share other strategies.

8. Leave the numbers on the number line from one day to the next so that students can look at the decimals relative to other decimals with which they have worked.

Scaffold for Variation 4:

- Use accessible fractions (e.g., fourths and eighths or thirds and sixths).

Extensions for Variation 4:

- Use a label other than 1 for the right-hand extreme (e.g., 2 or 3).
- Include mixed numbers and improper fractions in the numbers you place on the index cards.

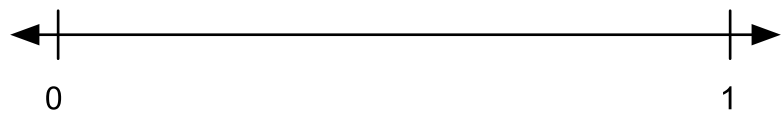
Guiding questions for Variation 4:

- Support your placement with a mathematically convincing argument.
- Name a fraction less/greater than yours. Prove it on the number line.

- How do you know your fraction is less/greater than one half?
- Name another fraction equivalent to yours.

<u>Variation 5</u>: Integrating Decimals, Fractions, And Percents

1. Label the extremes with “0” and “1.”



2. Write different decimals, fractions and percents on index cards.
   Use numbers such as 1/8, 0.125, 1/4, 0.25, 25%, 30 %, 1/3, 3/8, 0.375, 1/2, 0.5, 0.500, 50%, 5/8, 0.625, 2/3, 3/4, 6/8, 0.75, 75%, 7/8, 0.875, 100%.

3. Choose only 2 or 3 numbers to work with each time you do this routine. Make multiple copies of the same numbers. Give a card to each pair of students. Give them time to discuss where their number would make sense on the number line.

4. Have one pair of students place their card where they think their number belongs on the number line. Students must give a mathematically convincing argument as to why they are placing the number at this location.

5. Students discuss with their partners whether they agree or disagree with the placement of the card and why.

6. Class asks clarifying questions to the pair in the front of the room.

7. Students share other strategies.

8. Leave the numbers on the number line from one day to the next so that students can look at the amounts relative to other numbers with which they have worked.

Extensions for Variation 5:

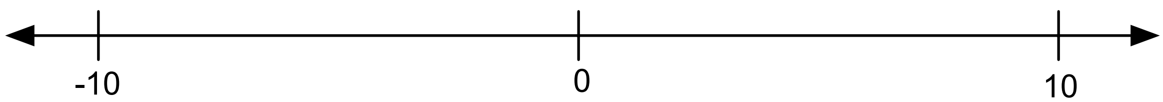
- Integrate more complicated values such as thirds, sixths and other repeating decimals.

Guiding Question for Variation 5:

- Support your placement with a mathematically convincing argument.
- What is another name for the value of your number?
- Name a number that is greater than your number. Prove it on the number line.
- Name a number that is less than your number. Prove it on the number line.

Variation 6: Integers

1. Label zero somewhere in the middle of the number line and label the extremes –10 and 10.



2. Write integers on index cards. Use numbers such as –1, -5, 3, 6, -8, 2, etc.

3. Choose only 2 or 3 integers to work with each time you do this routine. Make multiple copies of the same numbers. Give a card to each pair of students. Give students time to discuss where their number would make sense on the number line.

4. Have one pair of students place their card where they think their number belongs on the number line.

5. Students discuss with their partners whether they agree or disagree with the placement of the card and why.

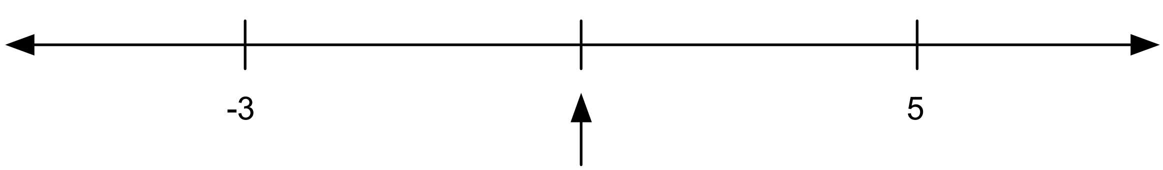
6. Class asks clarifying questions to the pair in the front of the room.

7. Students share other strategies.

8. Leave the numbers on the number line from one day to the next so that students can look at the amounts relative to other numbers with which they have worked.

Extensions for Variation 6:

- Write integers on the outlying marks such as –3 and 5.
  Have the class decide the value for the mark in the middle.
  In this example the arrow is pointing to 1.



- Include numbers on the index cards that are less than the number marked on the left and are greater than the number marked on the right. For example, put –5 on a card. The students will place it an appropriate distance to the left of the –3 in the example. This will help students realize that each time we work with number lines, these number lines are just part of the infinite number line.

- Discuss relative distances between the numbers involved in the routine. (e.g., How far is it from –1 to 2?)

Guiding questions for Variation 6:

- Support your placement with a mathematically convincing argument.
- Name an integer that is greater than this number.
- How much greater? Prove it on the number line.
- Name an integer that is less than this number.
- How much less? Prove it on the number line.

<u>Variation 7:</u> Very Large Numbers

1. Label the extreme to the left with a zero. Label the extreme to the right with million, ten million, or hundred million.

2. Write large numbers (into the millions) on index cards.

3. Choose only 2 or 3 numbers to work with each time you do this routine. Make multiple copies of the same numbers. Give a card to each pair of students. Give students time to discuss where their number would make sense on the number line.

4. Have one pair of students place their card where they think their number belongs on the number line.

5. Students discuss with their partners whether they agree or disagree with the placement of the card and why.

6. Class asks clarifying questions to the pair in the front of the room.

7. Students share other strategies.

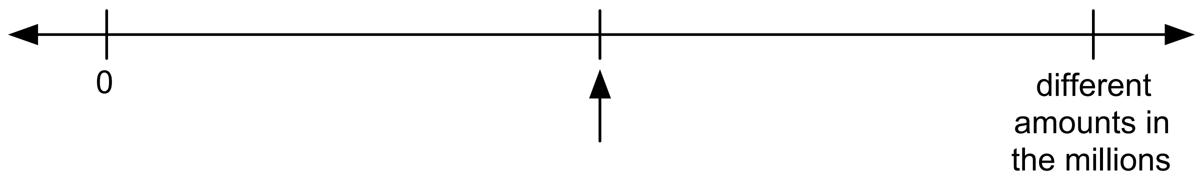
8. Leave the numbers on the number line from one day to the next so that students can look at the amounts relative to other numbers with which they have worked.

Scaffold for Variation 7:

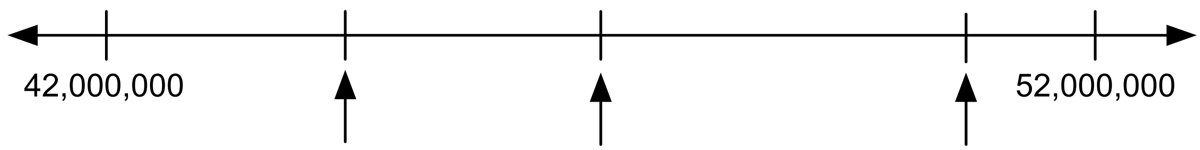
- Use more easily accessible numbers for the students to place (e.g., 4,000,000; 43,000,000; 625,000,000).

Extensions for Variation 7:

- Write zero on the mark to the left. Vary the million you write on the mark to the right. Have the class decide on a reasonable value for the mark in the middle.



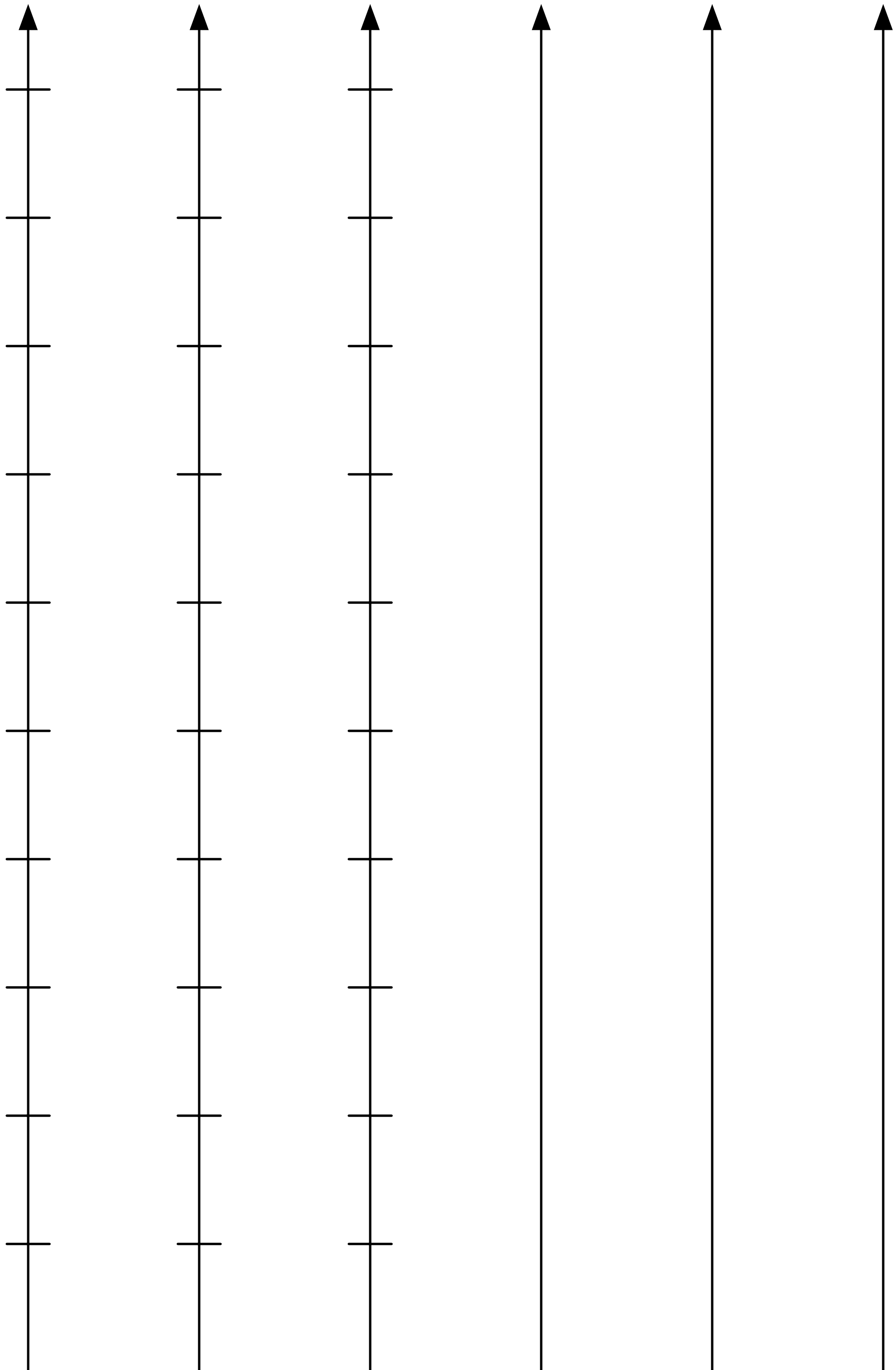
Change zero to a different value. This increases the rigor of labeling the unknown marks.



- Students place numbers that are relatively close together so that they must be more discriminating in their examination of their number (e.g., 42,721,000; 42,271,000; 42,172,000).
- Occasionally give students numbers that **don't** belong between the labeled marks (e.g., if you have labeled the outside marks "0" and "40,000,000, give students the number 53,531,671 to place. This reminds students that they are working with only **part** of the number line.)

Guiding questions for Variation 7:

- Support your placement with a mathematically convincing argument.
- Name a number that is greater than this number.
- How much greater? Prove it on the number line.
- Name a number that is less than this number.
- How much less? Prove it on the number line.
- What number is a hundred more/less than your number?
- What number is a thousand more/less than your number?
- What number is ten thousand more/less than your number?



MIDDLE LEVEL

Number Strings

Purpose:

- **To use number relationships to solve problems and to learn number facts**
- **To use known facts and relationships to determine unknown facts**
- **To develop and test conjectures**
- **To make generalizations about mathematical relationships, operations and properties**

Description:

This routine focuses on developing a sense of pattern and relationships among related problems. The task is at a higher level than merely recalling basic facts. Students identify and describe number patterns and relationships within and among equations. Students make conjectures about the patterns and relationships they notice. During this process, students explain their reasoning. Over time, students develop generalizations about important number relationships, operations and properties. These generalizations assist in solving problems and learning number facts.

Materials:

- Prepared list of number strings
- Whiteboard, chart paper, or overhead transparency
- Student journals, whiteboards, or scratch paper

Time: 10 minutes

Directions:

Example:

a. 2 x 5 =
b. 4 x 5 =
c. 8 x 5 =
d. 16 x 5 =
e. 32 x 5 =
f. 48 x 5 =
g. 48 x 50 =
h. 48 x 500 =
i. 48 x 0.5 =
j. 48 x 0.50
k. 48 x $\frac{1}{2}$ =
l. 48 = 2x
m. 24 x $\frac{x}{2}$ =
n. 48 x $\frac{1}{4}$ =
o. 48 = 4x
p. 12 x $\frac{x}{4}$ =

Directions continued:

1. Write equation "a" and ask students to solve mentally (e.g., 2 x 5 =). Equation "a" should be easily accessible to all students.

2. Have students check their answer with a partner.

3. Ask one student to share his/her solution with the class. Write the answer on the board to complete the equation (2 x 5 = 10).

4. Students show thumbs up or thumbs down for agreement or disagreement.
   - If there is agreement, go to equation "b".
   - If there is disagreement, then facilitate a class conversation around the strategies the student(s) used to arrive at the answer. Allow students to revise answers.

5. Give the students problem "b" to solve mentally (e.g., 4 x 5). Repeat, #2, #3, and #4 from above.

6. Write problem "c" (e.g., 8 x 5). Ask students how they could use what they know about the first two equations to solve this equation. Partner talk.

7. A volunteer shares his/her mathematical reasoning that derived an answer to this equation (e.g., "I know that the factor '2' in the first equation was multiplied by 2 to get the new factor '4' in the second equation. The '5' stayed the same, so the product was also multiplied by 2: 10 x 2 = 20. Since the '8' in the third equation is 2 four times and the '5' stayed the same, then the product should be also multiplied four times: 10 x 4 = 40").

   Note: If students are having difficulty sharing relationships, ask questions such as the following:
   - *How are equations "a" and "b" alike?*
   - *How are equations "a" and "b" different?*
   - *Describe the relationship between the factors?*
   - *Describe the relationship between the products?*
   - *How can we use these relationships to predict the product for equation "c?"*

8. Write problem "d" (e.g., 16 x 5). Ask students to predict their answer to this problem. Students share their predictions with their partner and explain their thinking. Teacher writes predictions on the board.

9. A volunteer shares his/her mathematical reasoning that derived the answer to this equation. (e.g., "16 x 5 =☐" could lead to a discussion about quadrupling the "4 x 5 =☐ " equation or doubling the "8 x 5 =☐ " equation.)

10. Repeat steps 8 and 9 for equations "e," "f," "g," and "h."

    Note: When students get to an equation that does not necessarily follow the same pattern (e.g., doubling), the discussion should yield many different strategies. (e.g., "48 x 5 =☐ " could be solved by adding the products of 16 x 5 and 32 x 5, or by multiplying the product of 8 x 5 by 6, or by multiplying the product of 2 x 5 by 24, etc.)

11. When the string is completed, facilitate a conversation about how relating a known equation can help students solve unknown equations. Listen for what relationships students notice throughout the string and how students are able to extend patterns beyond the string you have written. Ask students to make statements about the patterns and/or relationships that helped them to complete the string.

12. Examine the "conjectures" that the students share. Ask questions such as:
    - *Will doubling one factor always result in a doubled product? How can you prove your conjecture?*
    - *Will this always work? How can you prove your conjecture?*

Scaffold

Begin with strings that grow in a predictable way and are easily accessible to all students.

Possible Number Strings:

2 x 5 =
4 x 5 =
8 x 5 =
16 x 5 =
32 x 5 =
48 x 5 =
48 x 50 =
48 x 500 =
48 x 0.5 =
48 x 0.50 =
48 x $\frac{1}{2}$ =
48 = 2x
24 = $\frac{x}{2}$
48 x $\frac{1}{4}$ =
48 = 4x
12 = $\frac{x}{4}$

1 x 10 =
2 x 10 =
3 x 10 =
6 x 10 =
6 x 20 =
6 x 200 =
6 x 0.2 =
6 x 0.02 =
6 x $\frac{1}{10}$ =
6 = $\frac{x}{10}$
6 x $\frac{1}{100}$ =
6 = $\frac{x}{100}$
6 = $\frac{x}{1000}$

1 x 12 =
2 x 12 =
3 x 12 =
6 x 12 =
8 x 12 =
8 x 1.2 =
8 x 120 =
8 x 121 =
10 x 12 =
10 x 1.2 =
10 x 120 =
10 x 121 =
2x = 12
3x = 12
6x = 12
6x = 120
60x = 120
6 = $\frac{12}{x}$
60 = $\frac{120}{x}$

3 x 7 =
30 x 7 =
30 x 70 =
0.3 x 7 =
0.3 x 0.7 =
0.03 x 0.7 =
0.03 x 0.07 =
3x = 21
30x = 21
300x = 21
0.3x = 21
0.03x = 21
$\frac{3}{10}$ x 7 =
$\frac{3}{10}$ x 70 =
$\frac{3}{100}$ x 70 =
$\frac{3}{100}$ x 700 =

12 ÷ 12 =
12 ÷ 6 =
12 ÷ 4 =
12 ÷ 3 =
12 ÷ 2 =
12 ÷ 1 =
12 ÷ $\frac{1}{2}$ =
120 ÷ $\frac{1}{2}$ =
12 ÷ $\frac{1}{10}$ =
12 ÷ $\frac{1}{20}$ =
12 ÷ $\frac{1}{3}$ =
120 ÷ $\frac{1}{3}$ =
120 = 2x
120 = $\frac{x}{2}$

36 ÷ 3 =
36 ÷ 6 =
18 ÷ 6 =
180 ÷ 6 =
180 ÷ 12 =
1800 ÷ 12 =
3600 ÷ 12 =
36 ÷ 12 =
3.6 ÷ 12 =
0.36 ÷ 12 =
6x = 18
6x = 180
60x = 180
60x = 18
6 = $\frac{18}{x}$
6 = $\frac{180}{x}$
6 = $\frac{x}{3}$
60 = $\frac{x}{30}$

8 ÷ 2 =
16 ÷ 2 =
32 ÷ 2 =
48 ÷ 2 =
48 ÷ 4 =
480 ÷ 4 =
484 ÷ 4 =
480 ÷ 40 =
4.8 ÷ 4 =
0.48 ÷ 12 =
12x = 48
6x = 48
60x = 480
60x = 48
6 = $\frac{48}{x}$
6 = $\frac{480}{x}$
6 = $\frac{x}{8}$
60 = $\frac{x}{80}$

14 ÷ 7 =
140 ÷ 7 =
280 ÷ 7 =
287 ÷ 7 =
280 ÷ 14 =
2800 ÷ 14 =
2814 ÷ 14 =
2.8 ÷ 14 =
0.28 ÷ 14 =
28 ÷ 14 =
14x = 28
7x = 280
70x = 280
0.7x = 280
7 = $\frac{28}{x}$
7 = $\frac{280}{x}$
7 = $\frac{x}{4}$
70 = $\frac{x}{40}$

<u>Generalizations to develop through these strings</u>:
Note to Teacher: Do not <u>tell</u> students these generalizations. Ask students to make conjectures first and then ask them to test their conjectures using three or more examples. If the conjectures always hold true, then the students can make "generalizations".

In multiplication, many strings begin by doubling one factor while leaving the other factor the same (e.g., 2 x 5 becomes 4 x 5). This always doubles the product accordingly (e.g., 2 x 5 = <u>10</u> becomes 4 x 5 = <u>20</u>). The Big Idea associated with this pattern is: **By whatever amount the factor is multiplied, the product will be multiplied by the same amount.**

In division, this relationship holds true with the dividend and the quotient as well. **As the dividend is doubled** (8 ÷ 2 becomes 16 ÷ 2), **the quotient is doubled accordingly** (8 ÷ 2 = <u>4</u> becomes 16 ÷ 2 = <u>8</u>).

The divisor has an inverse (opposite) relationship with the quotient. **As the divisor is multiplied by an amount, the quotient is divided by that same amount** (e.g., 36 ÷ 3 = <u>12</u> becomes 36 ÷ 6 = <u>6</u>).

Sometimes the pattern is predictable because a factor is being doubled over and over, so the product doubles over and over, as well. But then, the pattern may change (e.g., 8 x 5 = <u>40</u>, 16 x 5 = <u>80</u>, 32 x 5 = <u>160</u>, then <u>48 x 5 = ☐</u>).

In order to make sense of this situation, a student must understand the associated Big Idea: **Numbers are the sum of more than one quantity** (e.g., 48 = 16 + 32). **The Distributive Property** states that when a number is being multiplied by a particular factor, it is equivalent to multiplying the number by the parts that make up that factor [e.g., 48 x 5 = (16 x 5) + (32 x 5)].

This Big Idea can help students develop an understanding of the relationships among numbers that will aid them in finding unknown products by relying on known facts (see *Using Strings to Learn Multiplication Facts* below).

<u>Example</u>:
Because 48 = 16 + 32, and students already know what 16 x 5 and 32 x 5 are, they can derive 48 x 5 as follows:
48 x 5 = (16 x 5) + (32 x 5)
<u>240</u> 80 + 160

The Distributive Property also states that when a dividend is being divided by a particular divisor. (e.g., 2814 ÷ 14), it is equivalent to dividing the parts that make up that dividend by the same divisor and then adding the quotients [e.g., 2814 ÷ 14 = (2800 ÷ 14) + (14 ÷ 14)].

<u>Example</u>:
Because 2814 = 2800 + 14, and students already know that 2800 ÷ 14 = 200 and 14 ÷ 14 = 1, they can derive 2814 ÷ 14 as follows:
2814 ÷ 14 = (2800 ÷ 14) + (14 ÷ 14)
<u>201</u> 200 + 1

<u>Using Strings to Learn Multiplication Facts</u>
Strings can be helpful to assist students to learn their multiplication facts as they learn to see the relationships among the facts.

<u>Example</u>: If a student cannot remember 8 x 6, but knows 4 x 6, all the student has to do is double the product of 4 x 6 because 8 = 2 x 4.
4 x 6 = 24
8 x 6 = 48

<u>Example</u>: If a student cannot remember 8 x 6, but knows 2 x 6 and 6 x 6, all the student has to do is find the product of these two equations and then find the sum of the products because 8 = 2 + 6.
2 x 6 = 12 6 x 6 = 36 8 x 6 = 48

Guiding Questions:

- What pattern(s) do you see?
- What stayed the same? What changed?
- How did it change?
- How did knowing the answers to the first equation help you figure out the answer to the next equation?
- Does this always work? How do you know?
- How are equations “a” and “b” alike?
- How are equations “a” and “b” different?
- What is the relationship between the factors?
- What is the relationship between the products?
- How can we use these relationships to predict the product for equation “c”?
- What is the relationship between the dividends?
- What is the relationship between the divisors?
- How can we use these relationships to predict the quotient for equation “c”?
- What is the relationship between the quotients?

Mental Math Strategies

- Number Talks
- Concepts of Equality
- Thinking Relationally
- Math Trains
- How Far? How Do You Know?
- Close, Far, and In Between

MIDDLE LEVEL

Mental Math Strategies
Number Talks

Purpose:

To develop computational fluency (accuracy, efficiency, flexibility) in order to focus students' attention so they will move from:

- figuring out the answers any way they can to . . .
- becoming more efficient at figuring out answers to . . .
- using efficient strategies

Description:

The teacher gives the class an equation to solve mentally. Students may use pencil and paper to keep track of the steps as they do the mental calculations. Students' strategies are shared and discussed to help all students think more flexibly as they work with numbers and operations.

Materials:

- Prepared problems to be explored
- Chalkboard, white board, or overhead transparency
- Individual white boards or pencil and paper
- Optional: Interlocking cubes; base ten materials; decimal squares

Time: 8-10 minutes

Directions:

Example: 9.8 + 8.7

1. Write an expression **horizontally** on the board (e.g., 9.8 + 8.7).

2. Ask students to think first and estimate their answer before attempting to solve the problem. Post estimates on the board. This will allow you to see how the students are developing their number sense and operational sense.

3. Ask students to mentally find the solution using a strategy that makes sense to them. Encourage students to "think first" and then check with models, if needed. Have tools available to help students visualize the problem if they need them (e.g., base ten blocks; 100 grids; decimal squares).

4. Ask students to explain to a partner how they solved the problem.

5. While students are discussing their strategies, walk among the groups listening to the explanations. Find those strategies you want to call attention to for the whole class. Choose strategies for discussion that you might want other students to think about and possibly experiment with. For example, in the problem 9.8 + 8.7 you might see the following strategy and want other students to think about and possibly experiment with it:

   9.8 + 8.7 =
   8.7 – .2 = 8.5
   9.8 + .2 = 10
   10 + 8.5 = 18.5

6. Call on a student to <u>fully</u> explain the steps he/she followed to solve the problem.

7. Record the steps precisely as the student explains them to you. Ask clarifying questions as needed to ensure that you understand the flow of the child's thinking. Be explicit about the mathematics.

   - "Why did you subtract .2 from 8.7?"
   - "Does this strategy always work? How do you know?"
   - "What did you know about the number 8.7 that allowed you to do that?"

8. As time allows, ask other students to share different methods they used for solving the equation. Follow up on each strategy shared by asking similar questions to those included in step 7. Publicly record these methods as well.

9. It is very important to facilitate a discussion about how the different representations/strategies relate to each other and result in the same answer.

See the following examples:

<u>Example - Guiding the Share-Out:</u>

<u>Scenario 1</u>: 6.3 – 2.7

<u>Public Recording</u>

2.7 + .3 = 3.0
6.3 + .3 = 6.6
6.6 – 3.0 = 3.6

<u>Student's Explanation</u>

"I added the same amount to both numbers to keep the difference the same. I chose 0.3 because it makes the 2.7 into a "friendly" number to subtract."

Possible teacher response:

"You said you added 0.3 to both numbers. How does adding 0.3 to both numbers keep the difference the same? Use a model to convince me."

<u>Scenario 2</u>: 6.3 – 2.7

<u>Public Recording</u>	<u>Student's Explanation</u>
2.7 + **0.3** = 3.0 3.0 + **3.0** = 6.0 6.0 + **0.3** = 6.3 **0.3 + 3.0 + 0.3** = 3.6	"I added 0.3 to 2.7 which makes 3.0. Then, I added 3 more to make 6. Then I added 0.3 more to make 6.3. I added together all the numbers I used. The answer is 3.6.

Possible teacher response:

"So, you used an "adding up" strategy. How does adding numbers help to find the difference? Why did you choose to add the numbers that you did? How did you keep track of the numbers you added? Each strategy is different, yet each arrives at the same answer for 6.3 – 2.7. Why do you think this is so?"

Scaffold:

- When beginning *Number Talks*, make sure that the problems and quantities are accessible and within each child's zone of proximal development. The numbers must be accessible so that the students are solving the equations mentally.

- If you have students in your classroom who are performing at diverse instructional levels, select 3 different problems for students to solve at 3 different levels. Allow students to choose the problem which they will solve. Select problems with varying levels of difficulty so that all students have access to a problem and all students are working at a level that pushes them to their optimal level. For example:

  4.63 – 0.27 6.3 – 2.7 6.3 – 0.7

- As the students' flexibility, accuracy and efficiency improve, increase the rigor of the problems by adjusting the numbers or operations.

- Allow the students to document on paper their intermediate steps **<u>as</u>** they are solving the problem.

Test Prep:

Some children who understand many mathematical ideas do not fare well on a standardized test given in a multiple choice format. Often, children guess a “letter” rather than reasoning through the problem. To improve children’s test taking strategies while building number and operational sense, the following strategies are suggested:

- Pose a problem just as a problem would be posed with a “*Number Talk*.” For example:

It takes a machine 12 minutes to fill 200 bottles of soda. At this rate, how many minutes will it take the machine to fill 500 bottles of soda?

- Ask students to think about the problem in a way that makes sense to them. For example:

“It takes 12 minutes for 200 bottles. If I double the number of bottles, I’ll have to double the number of minutes. 400 bottles must be 24 minutes. 500 bottles is 100 more bottles. 100 is half of 200, so it must take 6 minutes for 100 bottles. So, 24 minutes and 6 more minutes is 30 minutes.”

or

“It takes 12 minutes for 200 bottles. 100 is half of 200, so it must take 6 minutes for 100 bottles. 500 bottles is five times more than 100 (5 x 100), so the time must be 5 times more (6 x 5), so it must take 30 minutes for 500 bottles.”

- Only after the children have thought about the problem, show them the A., B., C., and D. responses. Ask them to choose the answer that is closest to their thinking. For example:

  A. 25 minutes
  B. 28 minutes
  C. 30 minutes
  D. 40 minutes

- Ask students to publicly share the methods they used for solving the problem. When it fits the problem, facilitate conversations about the reasonableness of each choice (e.g., *“Why would A. not have been a reasonable choice? Why would D. not have been a reasonable choice?*).

- The important piece is that students take the time to think and reason about the problem before they choose an answer (or guess). This must be a habit for whenever they are confronted with a problem to solve. Using this format once a week beginning very early in the school year, could help students “break” the habit of guessing and assist in higher scores on standardized tests.

Notes about *Number Talks*:

A. Keep them short.

B. Encourage sharing and clarify students' thinking.

C. Teach intentionally.

- Start where your children are.
- Choose related sequences of problems.
- Chart the students' thinking so that it can be saved and referred to later.

D. Create a safe and supportive environment.

- Accept answers without praise or criticism.
- Allow students to ask questions of each other.
- Encourage students to listen to each other.
- Encourage students to self-correct.

E. Vary the *Number Talks* to meet the range of needs.

- Vary the sharing strategies used.

  Pair share
  Share whole group
  Explain someone else's strategy

- Vary the level of difficulty within a *Number Talk*.

  Use written problems
  Use word problems

- Record the students' thinking using correct notation on the board, on the overhead, or on chart paper.

F. Give students lots of practice with the same kinds of problems.

G. When planning or implementing a *Number Talk*, consider the following:

- How do students get their answers?
- Can students use what they know for related problems?
- How well can students verbalize their thinking?
- Are errors way off or are they reasonable?

H. The role of the teacher during a *Number Talk* is to facilitate and guide the conversation.

- The teacher purposefully chooses children to share strategies that will move the class toward computational fluency.
- The teacher asks questions that draw attention to the relationships among

strategies.

- **It is important to focus on the mathematics, not just the variety of strategies.** *Mathematically, why does the strategy work?*

Examples:

Division:

245 ÷ 7	829 ÷ 9	1 1/2 ÷ 1/4
0.45 ÷ 0.3	6/8 ÷ 1/4	2/3 ÷ 1.6
48 ÷ 1.2	2.4 ÷ 1/5	16,000 ÷ 2,000

Addition and subtraction of fractions:

1 – 3/5	1/4 + 1/2	4/6 – 1/3
3/4 + 1/2	1 3/4 + 1/2	1/4 + 2/4
4/6 – 1/3	3/12 + 1/4	5/8 – 1/2

Inequalities:

Greater than, less than, or equal to? 89 + 15 ☐ 85 + 19

Greater than, less than, or equal to? 89 x 15 ☐ 85 x 19

Greater than, less than, or equal to? 16 x 38 ☐ 18 x 36

Greater than, less than, or equal to? 32 x 18 ☐ 38 x 12

Integers:

-98 + (-97)	- 27 - (-63)	100 + (- 49)

Expressions for students who need support:

156 - 38	62 - 33	100 - 49
750 + 250	372 + 98	59 + 36
864 - 500	370 + 99	855-56
104 – 39	87 + 49	58 – 39
91 – 53	37 + 86	499 + 76
17 x 8	25 x 6	450 ÷ 45
20 x 4 x 2	15 x 30	16 x 5

MIDDLE LEVEL

Concepts of Equality

Purpose:

- To develop an appropriate conception of equality and the equal sign
- To develop understanding that the equal sign denotes the relation between two equal quantities (rather than a command to carry out a calculation)

Description:

- Students are engaged in a discussion about the meaning of the equal sign.
- The context of this discussion is true/false and open number sentences.
- The number sentences provide a focus for students to articulate their ideas and to challenge their conceptions.
- The discussions assist in developing ways of thinking and communicating that embody the principles of algebraic reasoning.
- Students articulate mathematical principles that often are not explored or stated.
- Students must justify the principles that they propose in ways that convince others, and they must recognize and resolve conflicting assumptions and conclusions.

Materials:

Purposely planned number sentences and open number sentences. The numbers selected should be easily accessible to students. The focus is on the meaning of the equal sign, not on practice of operations.

Time: 10 minutes

Pre-assessment:

Before beginning this series of routines, ask your students to complete the following on a half-sheet Of paper:

What number would you put in the box to make this a true number sentence?

$$8 + 4 = \square + 5$$

*The following information is for you. Do not discuss this problem with your students.
This problem was given to thirty typical elementary-grade classes. The responses were as follows:

	Response - Percent Responding			
	7	12	17	12 and 17
Grades 1 and 2	5%	58%	13%	8%
Grades 3 and 4	9%	49%	25%	10%
Grades 5 and 6	2%	76%	21%	2%

Falkner, Levi, & Carpenter,1999

This data suggests that many elementary school students have serious misconceptions about the meaning of the equal sign as a relation between two equal quantities. Many seem to interpret the equal sign as a command to carry out a calculation (the answer is…).

This misconception limits students' ability to learn basic arithmetic ideas with understanding and their flexibility in representing and using those ideas. This creates even more serious problems as they move to algebra.

Directions:

1. Engage students in a general discussion about true/false number sentences or what it means for a number sentence to be true or false. Provide examples asking whether the number sentence is true or false and how they know it is true or false. For example:

8 – 5 = 3	3 x 4 = 15	599 + 468 = 1,067

2. Once students are familiar with true/false number sentences, equations can be introduced that may encourage them to examine their conceptions of the meaning of the equal sign.   Pose one equation at a time and lead a discussion as to whether the equation is true or false. Students must justify their claims. Do not tell. Lead a discussion and ask questions. For example:

4 + 5 = 9	9 = 4 + 5	9 = 9
4 + 5 = 4 + 5	4 + 5 = 5 + 4	4 + 5 = 6 + 3
3 x 4 = 12	12 = 3 x 4	12 = 12
3 x 4 = 3 x 4	3 x 4 = 4 x 3	3 x 4 = 2 x 6
15 – 7 = 8	8 = 15 – 7	8 = 8
15 – 7 = 15 – 7	15 – 7 = 7 - 15	15 – 7 = 16 – 8
24 ÷ 2 = 12	12 = 24 ÷ 2	12 = 12
24 ÷ 2 = 24 ÷ 2	24 ÷ 2 = 2 ÷ 24	24 ÷ 2 = 36 ÷3

Many of the examples above do not follow the familiar form with two numbers and an operation to the left of the equal sign and the answer to the right of the equal sign. This may throw some students into disequilibrium. Asking students to choose whether each number sentence is true or false can encourage them to examine their assumptions about the equal sign.

Note: We are trying to help students understand that the equal sign signifies a relation between two numbers. It is sometimes useful to use words that express that relation more directly (e.g., "Nine is the same as 4 plus 5").

3. Including zero in a number sentence may encourage students to accept a number sentence in which more than one number appears after the equal sign. For example:

9 + 5 = 14	9 + 5 = 14 + 0	9 + 5 = 0 + 14	9 + 5 = 13 + 1

4. Open number sentences given after the corresponding true/false questions are a nice way to follow up on the ideas that came out of the discussion of the true/false number sentence. The question being asked is:

"What number can you put in the box to make this number sentence true?"

4 + 5 = □	9 = 4 + □	9 = □
4 + 5 = □ + 5	4 + 5 = □ + 4	4 + 5 = □ + 3
□ = 4 + 5	4 + □ = 9	
3 x 4 = □	12 = 3 x □	12 = □
3 x 4 = □ x 4	3 x 4 = □ x 3	3 x 4 = □ x 6
□ = 3 x 4	3 x □ = 12	
15 – 7 = □	8 = 15 – □	8 = □
15 – 7 = □ – 7	15 – 7 = □ – 8	□ = 15 – 7
15 - □ = 8		

24 ÷ 2 = □	12 = 24 ÷ □	12 = □
24 ÷ 2 = □ ÷ 2	24 ÷ 2 = □ ÷ 3	□ = 24 ÷ 2
24 ÷ □ = 12		

Scaffolding:

The following are benchmarks to work toward as children's conception of the equal sign evolves.

1. Getting children to be specific about what they think the equal sign means (even if their thinking is incorrect). To do this, the conversation must go beyond just comparing the different answers to the problem. For example, in the problem 8 + 4 = □ + 5, some children might say:

    The equal sign must be preceded by two numbers joined by a plus or a minus and followed by the answer (resulting in an answer of 12 to this problem).

    You have to use all the numbers (resulting in an answer of 17 to this problem).

    Though this understanding is not correct, the articulation of conceptions represents progress.

2. Children accept as true some of the number sentences that are not of the form a + b = c (e.g., 8 = 5 + 3; 8 = 8; 3 + 5 = 8 + 0; or 3 + 5 = 3 + 5).

3. Children recognize that the equal sign represents a relation between two equal numbers (rather than "the answer is"). Children might compare the two sides of the equal sign by carrying out the calculation on each side.

4. Children are able to compare the mathematical expression without actually carrying out the calculation. For example: 8 + 4 = □ + 5

    A child might say, "*I saw that the 5 over here (pointing to the 5 in the number sentence) was one more than the 4 over here (pointing to the 4 in the number sentence), so the number in the box had to be one less than the 8. So it's 7.*"

Guiding Questions:

- *Why do you think that?*
- *Why do you think that you cannot write number sentences that look like that?*
- *Do you agree or disagree with Student A? Why?*
- *Why do you believe this equation is true?*
- *Why do you believe this equation is false?*
- *What do you do when there is more than one number that follows the equal sign?*
- *How do you know that the number that you put in the box makes the number sentence true?*
- *How can you figure out the number that goes in the box without doing any calculation?*

Extensions:

Pose problems that include rational numbers and integers.

Reference
Carpenter, Thomas P. Franke, Megan Loef, Levi, Linda, Thinking Mathematically: Integrating Arithmetic & Algebra in Elementary School, Portsmouth, N.H.: Heinemann, 2003.
Falkner, Karen P., Levi, Linda, & Carpenter, Thomas P. 1999. "Children's Understanding of Equality: A Foundation for Algebra." *Teaching Children Mathematics 6*, 232-236.

MIDDLE LEVEL

Thinking Relationally

Purpose:

- To make the learning of arithmetic richer
- To think flexibly about mathematical operations
- To compare mathematical expressions without actually carrying out the calculation
- To help students recognize without having to calculate that the expressions on each side of the equal sign represent the same number
- To provide a foundation for smoothing the transition to algebra

Note: In algebra, students must deal with expressions that involve adding, subtracting, multiplying, and dividing but that are not amenable to calculation (e.g., $3x + 7y – 4z$). They have to think about relations between expressions ($5x + 34 = 79 – 2x$) as they attempt to figure out how to transform equations in order to solve them.

Description:

Students are engaged in conversations about the relationships between numbers and how these relationships can be useful in finding solutions to problems. Students analyze expressions through the context of true/false and open number sentences. Students find ways to solve the problems by using number relations before calculating the answers.

Materials:

Purposely planned equations.
Note: Select equations that cannot be easily calculated. We want students to be motivated to look for relations. If equations can be easily calculated, the need does not exist to look for number relations.

Time: 10 minutes maximum

Directions:

1. To successfully implement relational thinking routines, the following classroom norms must be established:

- Students explain their thinking
- Students listen to one another
- Alternative strategies for solving a given problem are valued and discussed
- Solutions that involve more than simply calculating answers are not only accepted but valued

2. You, the teacher, will need to make decisions based on the needs of your class. As you select problems:

- Start with relatively easy problems and selected problems that provide an appropriate level of challenge based on what you have observed students doing on previous problems.
- Select problems that will challenge students but not be too difficult for them.
- Make decisions about what problems to use next based on students' responses to problems that they had already solved.

3. **<u>Goal 1</u>: For students to recognize that they do not always need to carry out calculations; they can compare expressions before they calculate.**

   Engage students in a general discussion about what it means for a number sentence to be true or false. Pose the following true/false problems (one at a time):

   $$12 - 9 = 3$$
   $$34 - 19 = 15$$
   $$5 + 7 = 11$$
   $$58 + 76 = 354$$

   Students explain how they know whether the number sentence is true or false.
   Students justify their solutions with their partner.
   Notice which students are calculating and which students are using relationships to determine whether the problems are true or false.

4. Pose the following true/false problem:

   $$27 + 48 - 48 = 27$$

   Students justify their answers.
   This problem establishes that students do not necessarily have to calculate to decide if a number sentence is true or false.

5. Ask students to see if they can figure out whether the following problem is true or false (without major adding or subtracting):

   $$48 + 63 - 62 = 49$$

   Students justify their answer.
   This problem extends the idea that was used in the previous problem.

6. Pose the following true/false problem:

   $$674 + 56 - 59 = 671$$

   Students justify their solutions.

   This problem is slightly more complicated than the preceding problem because students have to recognize that 59 breaks apart to 56 + 3 and that they can subtract 56 from the 56 given in the problem, and then they have to subtract 3 more from 674.

7. **<u>Goal 2</u>: To use properties of numbers and operations to think about relations between numerical expressions.**

   Review open number sentences. Pose the following problem:

   What number would you put in the box to make this a true number sentence?

   $$7 + 6 = \square + 5$$

   Students justify their solutions.

8. Pose the following problems (one at a time):

43 + 28 = ☐ + 42
28 + 32 = 27 + ☐
67 + 83 = ☐ + 82

Students justify their solutions.
Children must recognize that they can use relational thinking to solve these problems without carrying out all the calculations.

9. Up until this point, boxes have been used to represent an unknown in an open number sentence. Students readily adapt to using letters to represent variables and unknowns. Pose the following problem:

12 + 9 = 10 + 8 + c

What is the value of c?

Students justify their solutions.
If students justify their answers with an explanation focusing on computation, ask how this problem could be solved without adding 12 + 9 or 10 + 8 (e.g., 10 is two lwss than 12 and eight is one less than nine, so c must be 3).

10. Pose a problem with larger numbers but the same general structure, as follows:

345 + 576 = 342 + 574 + d

What is the value of d?

Students justify their solutions.

11. Pose the following problem:

46 + 28 = 27 + 50 – p

What is the value of p?

Students justify their solutions.

12. When students have figured out how to deal with addition problems, move to subtraction problems. Pose the following problem:

86 – 28 = 86 – 29 – g

What is the value of g?

13. **<u>Goal 3</u>: Using relational thinking to learn multiplication facts**

The following problems can be used to draw children's attention to relations among numbers that can make learning number facts easier.

- Knowing that addition and multiplication are commutative reduces the quantity of number facts that children have to learn by almost half.

True/false:

6 x 7 = 7 x 6

What number would you put in the box to make this a true number sentence?

4 x 8 = 8 x ☐

- Understanding the relation between addition and multiplication makes it possible for students to relate the learning of multiplication facts to their knowledge of addition.

True/False

3 x 7 = 7 + 7 + 7
3 x 7 = 14 + 7
4 x 6 = 12 + 12
6 x 4 = 4 + 4 + 4 + 4

- Focusing on specific relationships among multiplication facts can make it possible for students to build on the facts they have learned.

3 x 8 = 2 x 8 + 8
6 x 7 = 5 x 7 + 7
8 x 6 = 8 x 5 + 6
7 x 6 = 7 x 5 + 7
9 x 7 = 10 x 7 - 7

Sample problems to assist in developing relational thinking:

True/False (not all are true)

37 + 56 = 39 + 54

33 - 27 = 34 – 26

471 – 382 = 474 – 385

674 – 389 = 664 – 379

583 – 529 = 83 – 29

37 x 54 = 38 x 53

60 x 48 = 6 x 480

5 x 84 = 10 x 42

64 ÷ 14 = 32 ÷ 28

42 ÷ 16 = 84 ÷ 32

Sample problems for developing understanding of the properties of numbers and operations within numerical expressions:

73 + 56 = 71 + d

68 + b = 57 + 69

96 + 67 = 67 + p

87 + 45 = y + 46

92 - 57 = g - 56

56 - 23 = f – 25

74 - 37 = 75 - q

73 + 56 = 71 + 59 – d

68 + 58 = 57 + 69 – b

96 + 67 = 67 + 93 + p

87 + 45 = 86 + 46 + t

92 - 57 = 94 – 56 + g

56 - 23 = 59 – 25 – s

74 - 37 = 71 – 39 + q

Sample problems for developing base ten concepts:

True/False (not all are true)

56 = 50 + 6

87 = 7 + 80

93 = 9 + 30

47 + 38 = 40 + 7 + 30 + 8

24 + 78 = 78 + 20 + 2 + 2

63 - 28 = 60 – 20 – 3 – 8

94 = 80 + 14	63 - 28 = 60 – 20 + 3 – 8
94 = 70 + 24	0.78 = .078
246 = 24 x 10 + 6	1.95 = 1.9500

Reference
Carpenter, Thomas P. Franke, Megan Loef, Levi, Linda, Thinking Mathematically: Integrating Arithmetic & Algebra in Elementary School, Portsmouth, N.H.: Heinemann, 2003.

MIDDLE LEVEL

Mental Math Strategies

Math Trains

Purpose:

To develop fluency with mental calculations by thinking and reasoning with numbers and operations (NOT ORDER OF OPERATIONS).

Description:

Slowly dictate a series of numbers and operations to the class. Students will mentally calculate the result of each operation and apply to the next part of the number train until the end of the train is reached. Students will share their answers and discuss their strategies for mental calculations.

Materials: Prepare and write several trains before Routine time.

Directions:

1. Slowly call out the train, pausing after each number. During this time, the students perform the calculations mentally. [e.g., 44 x 2 (pause) – 30 (pause) + 8 (pause) – 8 = ?].
2. After calling out the whole train, give students the signal to "Share with your partner".
3. Students share their answer with their partner.
4. Ask, "Can anyone tell me what the answer might be?"
5. Write it on the board and then ask: "Does anyone have a different idea about the answer?" (also to be written on the board).
6. After recording all volunteered answers, write the dictated sequence on the board step by step. The students compute each step as a class. Ask for explanations for how the students solved that step.

Note: As you write the pieces of the sequence here, ignore the order of operations since this is verbal activity.

Scaffold:

- Start with short trains that include operations appropriate for your grade level and students. Watch your students' faces for indications of whether they need more support or are being appropriately challenged; adjust your wait time accordingly.
- Some students may need to write down the answers to the steps as they are calculating them.

Extensions:

Over time, adjust the number and type of operations in the trains. (e.g., +, -, x, ÷, %, powers, etc.)
Over time, adjust the numbers used (e.g., whole numbers, decimals, fractions, etc.).

Cautions:

- Don't adjust the operations **and** size of numbers at the same time. Decide which aspect you want to strengthen (operations or size of numbers) at a particular time and emphasize it.
- It is recommended that your students have had previous experiences with number talks to build their flexibility in thinking and reasoning with numbers and operations.
- Take your cues for wait time, type and quantity of operations, and the numbers used from your students. The reason for this routine is to help **all students** develop fluency with mental math—which includes accuracy, efficiency and flexibility with numbers and operations. This is not a race to find who is the fastest student to answer.

Examples:

(Ignore order of operations in the examples because this is a verbal, not written, activity. Read the trains from left to right.)

- 3/4 + 1/2 – 1 =
- 2^3 – 4 x 2 =
- 10% of 50 + 5 ÷ 2 =
- 0.2 x 5 +32 =
- 3^2 x 2^2 – 6 ÷ 2 =
- 50% of 30 – 3 ÷ 4 =
- 0.25 x 4 + 7 =

For those students who need more support:
- 4 x 2 – 3 + 4 – 1 =
- 15 ÷ 3 + 2 x 4 =
- 17 – 3 ÷ 2 =
- 16 + 2 ÷3 x 3 =
- 40 – 20 ÷ 5 x 2 =

- 100 ÷ 10 –3 x 6 =

Extension:
- Have students write their own trains. Collect and use them with the class.

MIDDLE LEVEL

Mental Math Strategies

How Far? How Do You Know?

Purpose:

To increase computational fluency and mathematical reasoning through:
- Using tenths and hundredths, find the difference between whole numbers
- Using landmark numbers and skip counting to solve problems
- Developing mental math strategies for addition and subtraction of integers
- Using the result from one problem to solve an unknown problem.

Description:

Students use tenths, hundredths, and other landmark numbers (i.e., 0.25, 0.5, 0.75) to find differences between numbers and integers.

Materials:

- 0 to 1 number line (on transparency)
- Student journals/ note books

Directions:

Variation 1:

1. Display a 0 to1 number line on the overhead.
2. Give students a beginning number (e.g., 0.42) and ask them how far it is from that number to 1.
3. After students have determined the difference, ask students to share with a partner their strategy for getting to 1. While students are sharing their strategies, walk among the groups, listening for strategies students used in which they used tenths and other benchmarks to get to 1. For example: “I thought it was 0.08 from 0.42 to 0.5 then 0.5 more

to 1. 0.08 + 0.5 = 0.58." The idea you want to strengthen is to find the difference between two numbers, it is advantageous to use an "add up" strategy that utilizes landmark numbers.
4. Ask 1 or 2 of those students to share those strategies with the class as you record them for the class to see.
5. Repeat with another number and facilitate the class conversation around these strategies.

<u>Variation 2</u>:

1. Give students a beginning number between 1 and 2. (e.g., 1.37) Ask them how far it is from that number to 2.
2. After students have determined the difference, ask students to share with a partner their strategy for getting to 2. While students are sharing their strategies, walk among the groups, listening for strategies students used in which they used tenths and other landmarks to get to 2.
3. Ask 1 or 2 of those students to share those strategies with the class.
4. Record their strategies.
5. Repeat with other numbers.

Scaffold:
If students are struggling to find decimal differences from a number to the next whole number, take some time to work with whole numbers and help them make connections between decimal amounts from 0 to 1 and whole numbers from 1 to 100. Ask them to take a number and find out how far it is from 100 (e.g.; "How far is 68 from 100?" "When I think about it, I said that it was 7 more to 75 and I know its 25 more to 100."). Then move on to 1000's etc. (e.g., How far is 86 from 1000? "I thought it was 4 from 86 to 90 and 10 from 90 to 100. Then it's another 900 to 1000. So 4 + 10 + 900 = 914." **Or** How far is 721 from 1000? "I figured it is 4 to 725, 75 from 725 to 800 and 200 from 800 to 1000. So 4 + 75 + 200 = 279.")

<u>Variation 3</u>:

1. Give students a beginning number between -3 and 3. (e.g., -1) Ask them how far it is from that number to 3.
2. After students have determined the difference, ask students to share with a partner their strategy for getting to 3. While students are sharing their strategies, walk among the groups, listening for strategies students used. Share out and record strategies.
3. Give students a beginning number between –3 and 3 such as -2.25. Ask them how far it is from 3. Repeat step #2.
4. Repeat with other numbers.

Note:
- In any variation, ask students to finish their conversations by recording their strategy in their journals/note books.
- Adjust the quantities your students are working with to give them access to the strategies. Some students will need smaller numbers, but the strategies should be the same.

- It is important that students build mental models in their heads so they can visualize distances between number and integer amounts.

MIDDLE LEVEL

Mental Math Strategies

Close, Far and In Between

Purpose:

To increase an understanding of the relative magnitudes of numbers.

Description:

Students look at 3 numbers provided by the teacher to determine their relative proximity to each other and to other numbers.

Materials:

- Base 10 blocks (available for scaffolding)
- Fraction bars (available for scaffolding)
- Decimal grids (available for scaffolding)
- Number lines (available for scaffolding)

Directions:

1. Write 3 numbers on a transparency. (e.g., 1.2, 2.3, 3.2)

2. Students will work in pairs to answer some of the following questions. There are too many questions to ask at each routine time. Vary the questions you ask. Have several models/pictorials available to help students make meaning of the numbers given.

   - Which two are closest? How do you know?
   - How did you think about the differences between the numbers?
   - How do you know [a number] is larger/smaller than [another number]?
   - Which is closer to [a number]? By how much? (e.g., Which of the 3 numbers is closest to 1.8?)
   - Name 3 numbers between [one of the numbers] and [one of the other numbers].

(e.g., Name 3 numbers between 2.3 and 3.2.)

- How do you know [a number] is between the 2 numbers we named? Which number is it closer to? How do you know?
- What number is 1 larger than [one of the numbers]?
- What number is [say a decimal amount] smaller than [one of the numbers]? (e.g., What number is 0.4 less than 3.2?

- Name a number that is smaller than all three numbers.
- Name 2 numbers that make these seem large.
- When you saw [a number], how did you picture that amount in your head?

3. Discuss some of the answers the partners give.

Examples:

- $\frac{1}{4}$ $\frac{1}{2}$ $\frac{3}{5}$
- 0.03 0.16 0.111
- $1\frac{7}{8}$ $2\frac{1}{5}$ $1\frac{3}{8}$
- 2.007 2.714 2.417

Extension: Include integers.

- -4 3.1 1.3
- 1.4 -2.15 2.55

Number Translations

MIDDLE LEVEL

Number Translations

Purpose:

To develop a sense of quantity by representing fraction, decimal, and percent amounts in multiple ways and making connections among representations.

Description:

Students translate fractional amounts into multiple representations.
These representations include:

- Pictorial models
- Word form
- Decimal
- Fraction
- Percent
- Number line

Materials:

- Number Translations blackline master
- Large 10 x 10 grid (or overhead transparency)
- Small 10 x 10 grids
- Chart paper

Time: 10 minutes

Directions:

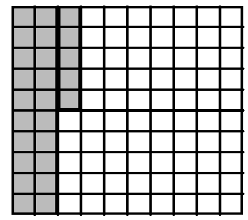
1. Choose a fractional amount.

2. Write it in one of the word forms on the chart paper (e.g., two-eighths).

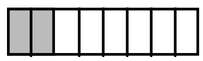
3. Working in any order, have students translate the written fractional amount into all the forms listed on the *Number Translations* sheet:

   For example: two-eighths

- Students show that fractional amount on the 10 x 10 grid.



- Students draw a different pictorial representation of the amount.



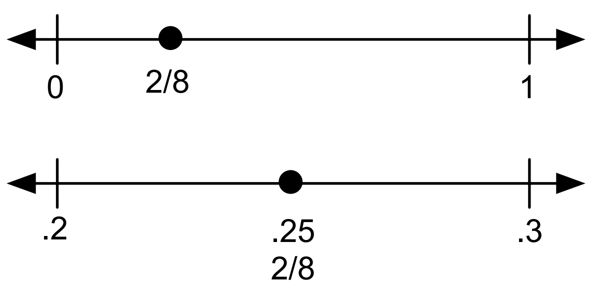
- Students write it in standard decimal form using symbols and words (e.g., 0.25, twenty-five hundredths).

- Students write it in fractional form using symbols and words (e.g., 2/8; two-eighths). Include simplest form if appropriate. (e.g., 1/4, one-fourth).

- Students write an equivalent decimal in symbols and words. (e.g., 0.250; two hundred-fifty thousandths).

- Students write it as a percent in symbols and words (e.g., 25%; twenty-five percent).

- Students locate it on 2 number lines using different scales (e.g., 0 – 1 and 0.2 – 0.3).



4. As you discuss each translation with the class, ask different students to record each translation on a class chart and post.

5. Starting with the second day of this routine, have students compare today's number with one or more of the previous amounts.

Note: Instead of continuing to run off the form for the students to use each day, have the students include all the data in a page in their student journals.

Caution: Some fractions cannot be easily represented on a 10 x 10 grid. Irrational numbers such as 1/7 and repeating decimals such as 1/3 cannot be represented accurately. In this case, students could shade in approximations.

Scaffold

- Start with whole group; move into partner or small-group work; and ultimately move into individual accountability.

- Use amounts with which students can make meaning (e.g., use benchmark numbers such as one-tenth or five-tenths).

Extensions

- Choose decimals that end at different decimal places (e.g., on different days give the students 0.7, 0.64, 0.355).

- Start with a different representation (i.e., instead of starting with the word form, start with the 10 x 10 grid, the decimal form, the fractional form or the number line).

Guiding questions:

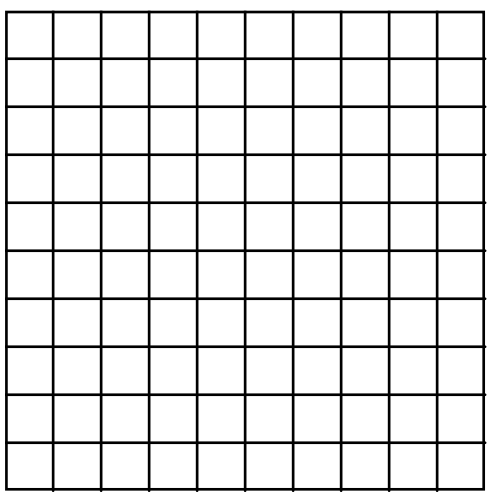
- Ask students to verify that the way they modeled/placed/wrote each representation is reasonable and makes sense.

- How are (name 2 of the representations) alike? How are they different (e.g., "How is the grid representation like the fractional form? How is it different?" or "How are the 2 number lines alike? How are they different?")?

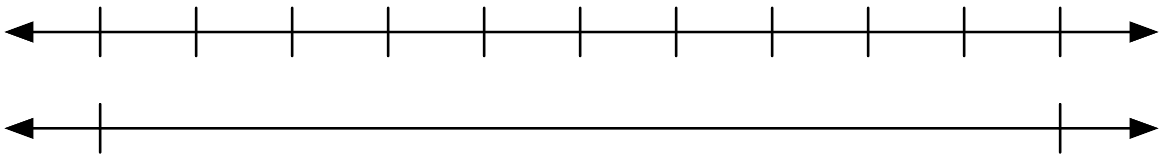
- How is today's number like Monday's number?

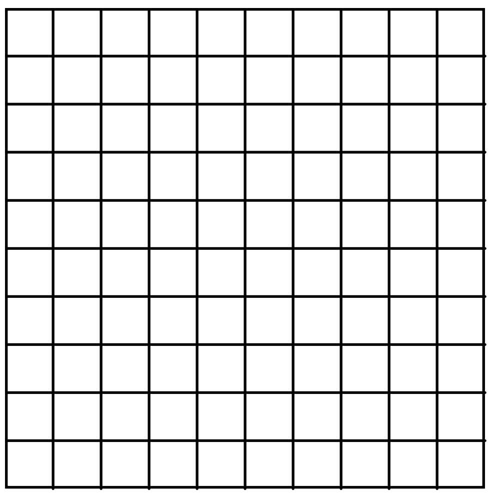
- Is today's number greater or less than Monday's number? How much greater/less?

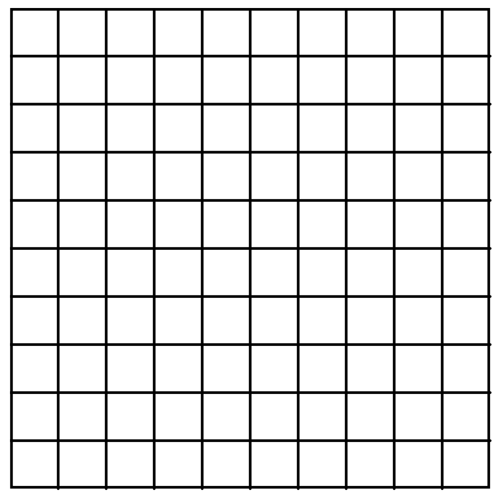
Number Translations

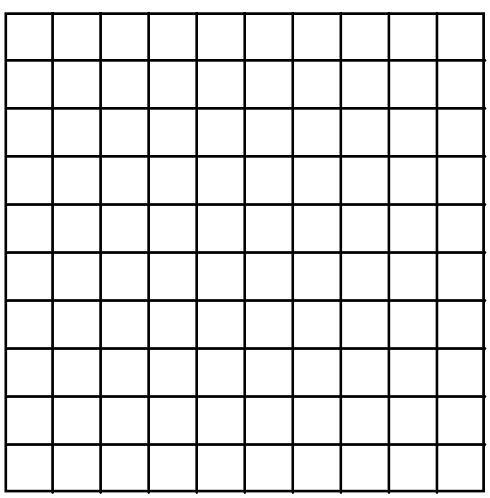
This grid is the whole. Shade in the fraction. 	Draw the fractional amount
Standard Decimal Form	Simplest Fractional Form
Standard Decimal Form in Words	Simplest Fractional Form in Words
Equivalent Decimal	Percent Form

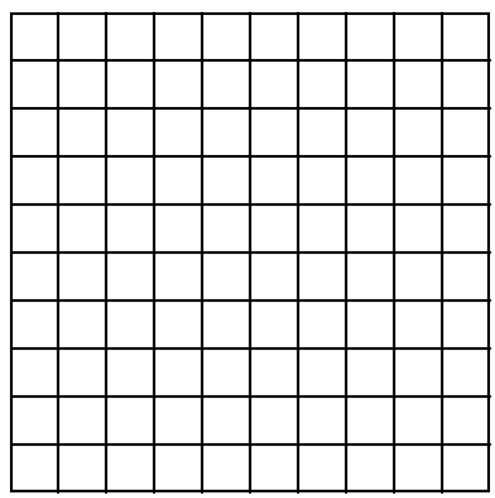
Equivalent Decimal in Words	Percent Form in Words

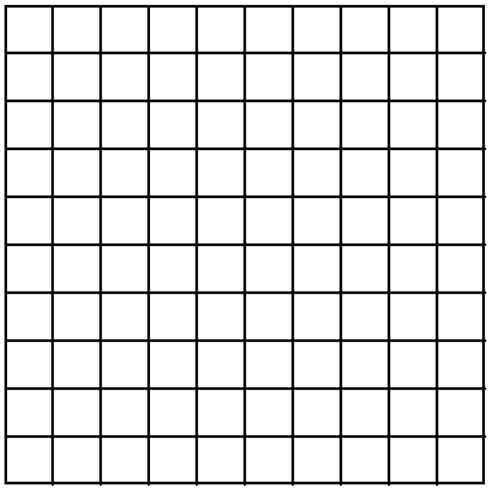
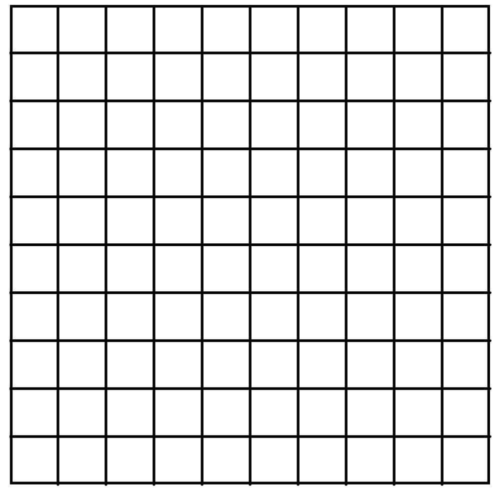










Silent Board Game

- Function Machines/In and Out Boxes

MIDDLE LEVEL

SILENT BOARD GAME

(Function Machines/In and Out Boxes)

Purpose:

Students recognize, describe, and generalize patterns and functional relationships.
A function is a relationship in which two sets are linked by a rule that pairs each element of the first set with exactly one element of the second set.

Description:

Students analyze a set of number pairs to determine the rule that relates the numbers in each pair. The data are presented in the form of a function table (T-table) generated from an "In and Out Box." Students will describe rules for relating inputs and outputs and construct inverse operation rules.

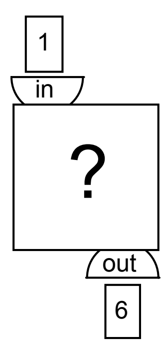
Materials:

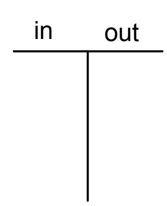
- T-table transparencies (What is the Rule?)
- "In and Out Box" transparency

Time: 8-10 minutes

Directions:

1. Display the "In and Out Box" on the overhead. Tell students that when you put a number in the box, a secret rule changes the number, and out comes a new number. Demonstrate with "1." You put "1" into the box, and "out" comes "6."





2. We organize the inputs and outputs with a T-table.

3. The job of the students is to determine the rule.

4. Write the first number in each column (e.g., "1" in the first column, "6" in the second column).

   Note: The example that we will be using may be complicated for early in the year. You may want to use a different "rule" when you introduce this routine to your class.

in	out
1	6

5. Ask pairs of students to discuss possible rules that could cause the number in the first column to become the number in the second column (e.g., add 5, multiply by 6, multiply by 3 and add 3).

6. Write the second number in each column (e.g., "2" in the first column, "6" in the second column).

in	out
1	6
2	9

7. Ask the students to check the rule they came up with for the first pair of numbers to see if the rule will apply to both sets of numbers. If the rule does not apply, ask students to think about a rule that could describe both sets of numbers.

8. Give students a third pair of numbers (e.g., 3 in the first column, 12 in the second). Does the rule still work? If not, students think about a different rule that could describe all three sets of numbers.

in	out
1	6
2	9
3	12

9. Ask students to write/discuss with their partner another pair of numbers that could fit their rule.

10. Ask for volunteers to share what they think other pairs of numbers could be that would fit the rule. Record these numbers on the transparency (without judgment as to whether it is correct or incorrect).

in	out
1	6
2	9
3	12
4	16
5	18
10	32
8	27

11. Partners discuss all the pairs of numbers on the transparency to see if they agree, or if there are inconsistencies in the pattern. Facilitate a class discussion about what operation(s) were working on each "in" to get each "out."

For Example:

For the teacher.
DO NOT TELL
→

1 + 1 + 1 + 3 = 6
3 x 1 + 3 = 6
3(1 + 1) = 6
2 + 2 + 2 + 3 = 9
3 x 2 + 3 = 9
3(2 + 1) = 9
3 + 3 + 3 + 3 = 12
3 x 3 + 3 = 12
3(3 + 1) = 12

in	out
1	6
2	9
3	12
~~4~~	~~16~~
5	18
~~10~~	~~32~~
8	27

Discuss similarities and differences, such as “Sue multiplied the first number by 3 and then added 3 more. But Julian added two 1’s to the first number then multiplied **that** number by 3. Why do these two rules still work?”

12. Based on the above conversation for each of the specific numbers, have partners discuss what they believe the rule is for changing the first number in the pair to the second number in the pair. <u>Students are now speaking in general terms, what must be done to any number, not just the numbers written on the table</u>.

    For the teacher: DO NOT TELL. In the example, each number in the “in” box is added to itself twice, plus three more; or, tripled, plus three more; or, multiplied by three, plus three more.

13. Ask for volunteers to share their rules. Record on the transparency the rules that students have generated. You will be recording the words, not numeric symbols (e.g., “The number in the ‘out’ box is equal to each number in the ‘in’ box multiplied by 3 plus three more”).

14. Discuss how the rules are the same. Discuss how the rules are different. Do the rules accurately apply to each of the pairs of numbers? Facilitate a class discussion about how the rules work and how each is a different representation of the same pattern.

15. Once the class agrees on rules that will work, ask the partners to figure out what other pairs of numbers would follow the rules.

16. Ask students to volunteer some of their pairs as you record them. Have students explain why each pair of numbers follows the rule.

17. Ask partners to “translate” the rules written in words to the same rule written with mathematical symbols. For example: The rule may be to multiply the first number by 3 and then add 3; or, add 1 to the first number and then multiply that number by 3. In mathematical symbols, this would be $3n + 3$ and $3(n + 1)$. These expressions are 2 forms of the same rule, just written in different ways.

Note: Remind students that when we speak about “any number” instead of a specific number, we represent “any number” with a variable such as “n”.

18. Give the students new "in" numbers. What would be the "out." How do you know?

Scaffold:

- Use smaller numbers and/or rules with only 1 operation (e.g., multiply by 2; add 5).

Extensions:

- Include rules with more than one step, as in the above example (e.g., multiply by 2, then subtract 1).

- Use fractions or decimals as the numbers in the first column.

in	out
.25	.5
.5	1
1	2

in	out
1/2	1
1/4	1/2
1/8	1/4
1/16	1/8

- Give the students the first pair of numbers as explained above. Then, give the "out" number in the set. Ask students to find the "in" number in the set.

in	out
1	5
3	7
?	9
n	n+4
n-4	n

Note: Working backwards provides opportunities for using inverse operations to find missing elements and writing rules. For example: The In-Out rule is "n + 4;" the Out-In rule is "n - 4".

SILENT BOARD GAME: Once students are familiar with the routine, place the first pair of values in the table. Put additional values for either In or Out and offer a marker to a student volunteer to fill in the resulting value. No one is allowed to talk! Students indicate their agreement or disagreement with thumbs up or thumbs down. Once all blanks have been filled, students are invited to volunteer to write the rule both algebraically and in words.

Examples:

In	Out

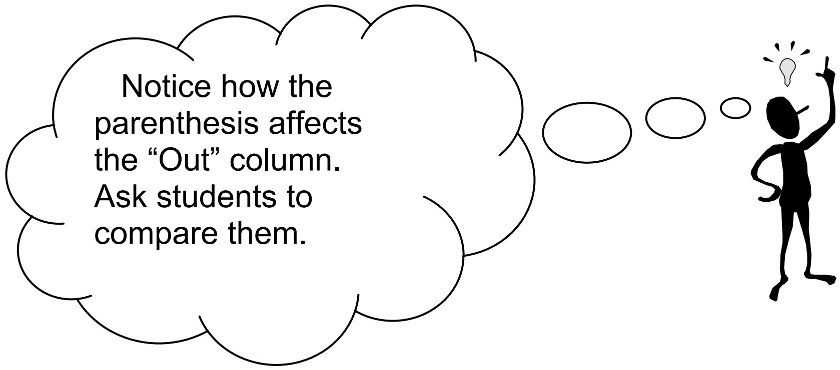
In	Out

In	Out

0.3	1.9
0.5	2.5
0.9	3.7
n	3n+1

1/2	3
1	4
1 1/2	5
n	2(n+1)

1/2	2
1	3
1 1/2	4
n	2n + 1



In	Out
1	1
5	25
3	9
n	n x n or n^2

In	Out
1/3	3
1	5
2/3	4
3	11
n	3n + 2

In	Out
4	12
2	6
5	15
8	24
n	3n

In	Out
3	4
1/2	1 1/2
17	18
1.25	2.25
n	n + 1

In	Out
25	24
1	0
-3	-4
0	-1
n	n - 1

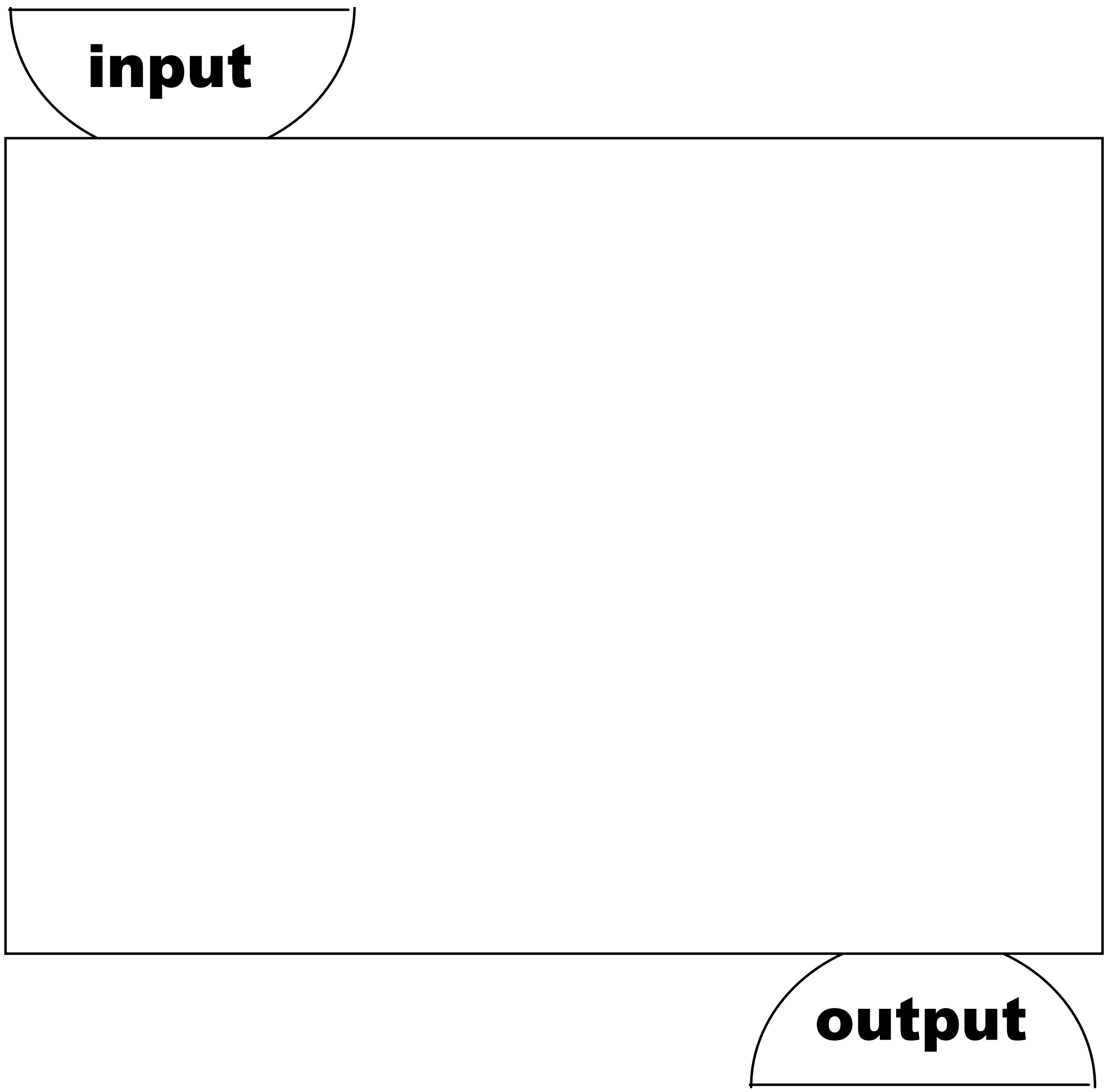
In	Out
3	5
10	19
5	9
1	1
n	2n - 1

In	Out
1	3
2	5
3	7
n	2n + 1

In	Out
5	15
12	22
20	30
0	10
n	n + 10

In	Out
-4	1
-2	3
0	5
5	10
n	n + 5

FUNCTION MACHINE


input
output

In	Out

What is the rule?