Developing Algebraic Thinking through Pattern Exploration

LESLEY LEE AND VIKTOR FREIMAN

PATTERN EXPLORATION IS A PIVOTAL ACTIVITY in all mathematics, indeed in all the scientific disciplines. Children who are attempting to express perceived patterns mathematically are in an excellent position to learn algebraic language and engage in algebraic activity. *Principles and Standards for School Mathematics* (NCTM 2000) acknowledges the relationship of pattern exploration and algebraic thinking by placing pattern work within the Algebra strand. Yet one can undertake considerable pattern exploration without engaging students in any algebraic thinking whatsoever and teachers may, themselves, be unclear about how patterns can be used to further algebraic thinking. Work with repeating patterns in the early grades, or teaching patterns as a "topic" in the middle grades, may not foster the development of algebraic thinking in students. In this article, we will address this question: How can teachers exploit pattern work to further algebraic thinking and introduce the formal study of algebra in middle school?

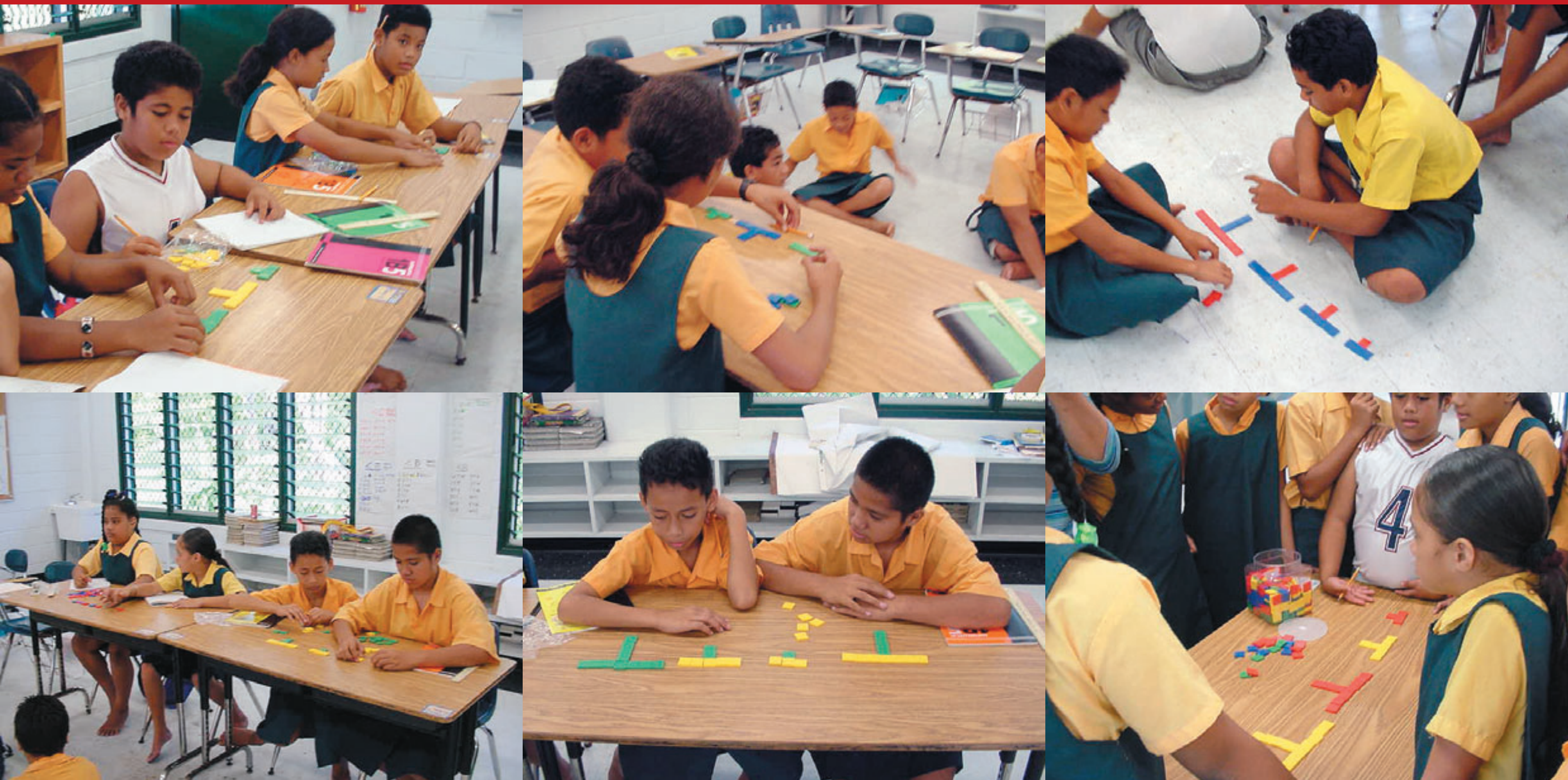
LESLEY LEE, *leel@prel.org, works as a mathematics specialist in Honolulu and the Pacific Islands for Pacific Resources for Education and Learning (PREL). Before coming to PREL, she worked in research and mathematics education in Montreal, Quebec.* VIKTOR FREIMAN, *freimanv@umoncton.ca, now a professor at the University of Moncton in New Brunswick, taught mathematics at Marie Claire Academy in the Montreal area, where the classroom research took place that inspired this writing.*

An Exemplary Pattern

FOR ILLUSTRATIVE PURPOSES, WE WILL REFER TO the "growing T" pattern (see **fig. 1**). There are a number of reasons to focus on this type of pattern.

- Seeing a pattern is a necessary first step in pattern exploration, and simple growing patterns are explicit, visual, and easy to see. On the one hand, although repeating patterns such as *box, box, circle, box, box, circle* are explicit, it is harder to see how they can be exploited for algebraic ends. They are, on the other hand, very valuable in the early grades for working with the idea of unit or identifying the repeating unit. In our research with kindergarten children, we found considerable interference between repeating and growing patterns, where the latter were often treated as if they were the former.
- Using number patterns such as 5, 8, 11, 14, . . . are interesting, but they are less useful in the initial work of viewing a pattern because they are less visual and do not give rise to a multiplicity of "seeings."

Fig. 1 A growing T pattern



- Deciding how to start the pattern is important. Note that we did not start this pattern with a single star, for two reasons. First, in our research we have found a number of students who object to a single star being called a T. Second, the algebraic expression of the pattern beginning with a single star is slightly more complicated because the general expression involves subtraction ($3n - 2$ as opposed to $3n + 1$).

Algebraic Avoidance in Pattern Work

BEFORE ADDRESSING HOW TO EXPLOIT PATterns such as the growing T to enrich algebraic thinking, we would like to take a moment to examine some common practices in the Quebec grade 7 classrooms where we began our research. In the Quebec curriculum, the word "patterns" appears with a flag to indicate that this theme is considered to be preparation for algebra. Many teachers cover the topic very quickly, presenting only patterns that lead to $an + b$ expressions and sharing their trick of arriving at that expression early on. In the case of the growing T, teachers often turn it into a number pattern by immediately asking these questions: "How many stars are in the 1st T? The 2nd? The 3rd? It is increasing by how many each time?" Once the students have answered the last question, they are told that the number is the a in the expression $an + b$. To get the b, students need only examine the 1st T in the pattern. It is hard to imagine that any algebraic thinking could be developed through this approach.

Research shows that more misconceptions than conceptions arise from such a treatment of patterns (Lee and Wheeler 1987; Orton and Orton 1999; Stacey 1989). For instance, although students are able to provide the n formula fairly quickly, it appears to have little meaning for them. In the Lee and Wheeler (1987) research, none of the 354 grade-10 students tested had the reflex to substitute values for n to see if their formula was valid for the given configurations. In interviews, some students indicated that n was a "big" number. Moreover, when asked how many elements (or dots, in this case) composed the 57th figure, a number of students indicated that they would have to draw the figure to answer the question. For these students, the formula applied only to the nth figure, thought of as being somewhere out there, but not any specific figure and certainly not the 1st or 2nd or 57th.

What Are Some Good Questions to Ask about Patterns?

IF STUDENTS ENGAGE IN PATTERN WORK WITHout developing any algebraic thinking and, in the worst-case scenario, it leads to algebraic misconceptions down the road, then it is important to consider the topic carefully. Some questions that we have found particularly productive follow.

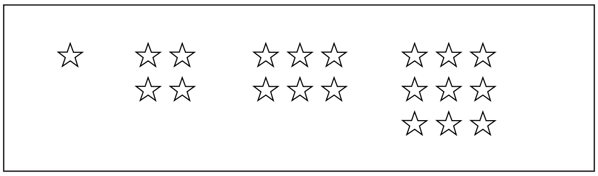


Fig. 2 Can you find the pattern?

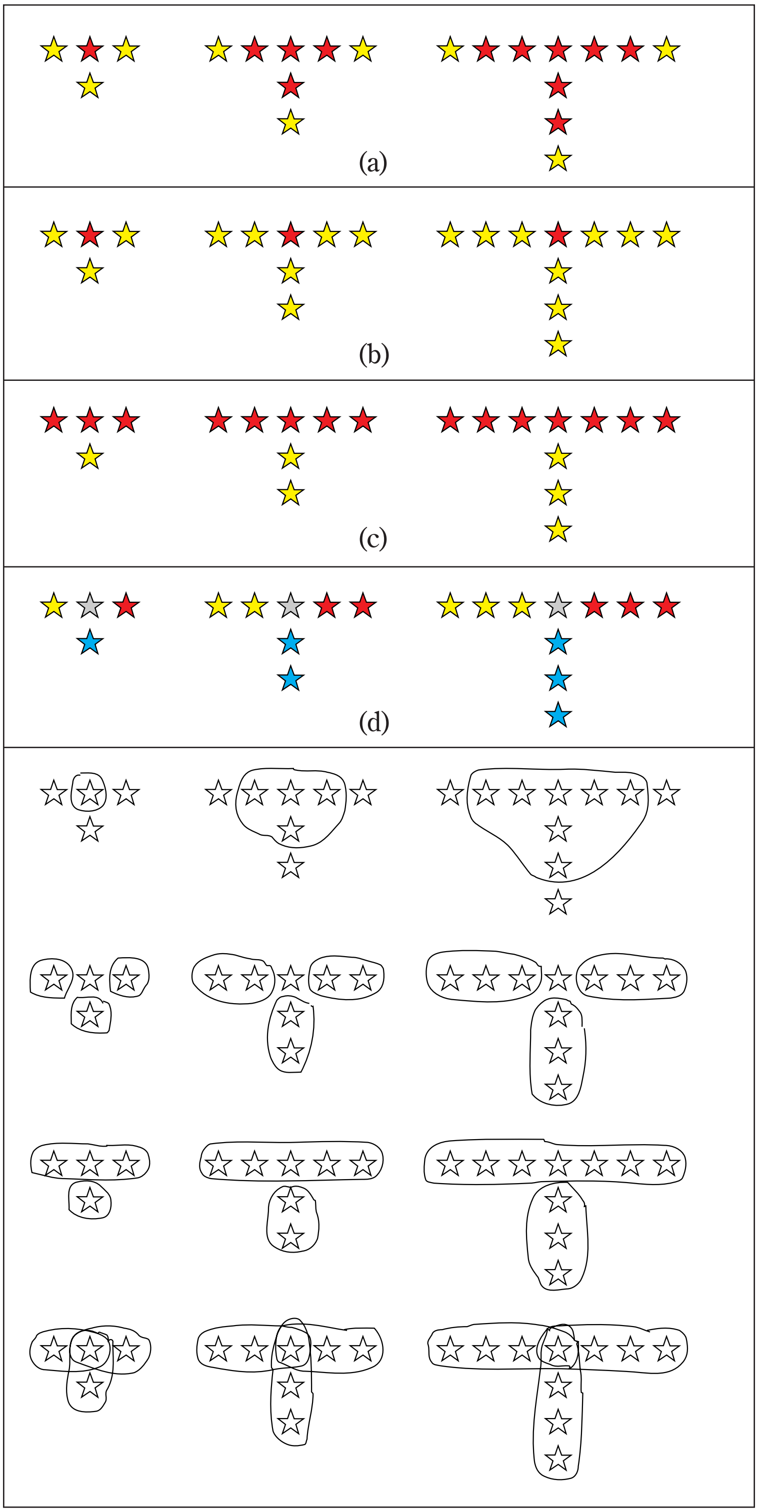


Fig. 3 Different "seeings," or ways that students saw a pattern. The bottom half of the figure shows other views.

1. In which of these drawings can you see a pattern?

This question is not often asked in classrooms. Since students are usually given configurations or number sequences that are patterns, they will work for a long time trying to impose a pattern on items that do not have one. It is easy to produce nonpatterns by simply dropping one figure from the first few pattern entries. A group of preservice teachers was recently presented with four configurations, the first of which was a growing T pattern. The second is illustrated in **figure 2**.

The main reaction to the diagram was to impose increasingly elaborate pattern rules (involving even and uneven numbered forms). A minority of the students were convinced that a mistake had been made in printing the pattern. None objected that this simply was not a pattern.

Although seeing a pattern is an important mathematical skill, noting a pattern where there is none can lead to mathematical disasters or quite simply be a huge waste of time (months, in the case of practicing mathematicians).

2. How many different patterns can you see in this drawing?

This step is crucial in using patterns to develop rich algebraic thinking. It also leads to multiple expressions of the same pattern, thereby opening the door to dealing with equivalent expressions and symbolic manipulation.

Seeing a pattern is not sufficient. Although young students express a wide range of pattern perceptions for a single configuration and show considerable flexibility in moving from one to the other, older students tend to grab onto one pattern perception and block out others for fear that multiple perceptions will "mix me up" (Lee 1996). Middle school students, with only a bit of encouragement, can be drawn into the pattern game of multiple perspectives and come up with a surprisingly long list of "seeings." **Figure 3** presents some of the ways that students saw the growing T pattern.

To elicit and share these multiple ways of seeing, some subquestions are suggested.

How do you draw the next figure? Students can be asked to draw the next figure on the board or overhead projector while the class is asked to carefully observe the order in which the student draws the elements of the figure. In the case of the 4th figure in the growing T, student (a) in **figure 3** copied the 3rd T and then placed a single star on each extremity; student (b) placed a central star and then added the three arms of four stars around it. A horizontal line of nine stars followed by a vertical line of four stars under the middle star (the fifth) of that line was the offering of student (c). Student (d) drew five stars horizontally, then added five more stars vertically and horizontally again, but erased one star each time when he realized they were overlapping on the central star.

Students can be asked to verbally describe one another's drawings and offer their own description if they are different. Each drawing of the 4th T expresses a very different view of the pattern.

How do you draw the 10th figure? After exhausting and sharing all the various views of the 4th and perhaps the 5th figure, students may be asked to draw the 10th T on the board. A few students may be able to draw it directly. However, most students will initially feel the need to draw the 5th through the 9th Ts before attempting the 10th, even if their pattern perception does not involve adding (here, three stars) to the previous figure. The next question will push students to find a quicker route.

How would you draw the 58th figure? Here we have moved into "saying" a pattern. Some students, like a few in our experimental class, would volunteer to actually draw the 58th T (and take up the entire mathematics period) to the despair of their classmates. However, most students understand that a description in words would suffice. Different descriptions of how to draw the 58th figure—corresponding to the different views elicited previously—should be encouraged. Other positions such as the 24th, the 100th, and the 201st can be the subject of questioning until the students appear ready for a more general statement.

How would you tell someone how to draw any figure at all? By this stage, you might expect to hear these ways to draw any T.

> "You draw a star and then add arms of that number to the right, the left, and below."
>
> "You take the T before it and add a star to each end."
>
> "You make the T out of three pieces, each one having the number plus one star. You overlap these pieces at the center of the T."
>
> "You double the number and take away one and that gives you the number of stars in the horizontal line. Then you draw the vertical part of the T under the middle star by adding the same number of stars as the number you were given."

How would you restate that description if I tell you that the figure is the n*th?* An example might be offered that corresponds to one of these statements: "Draw a star and then add arms of n stars to the right, the left, and below." Students can then be encouraged to reformulate their previous general statements in terms of n.

3. I have a box of twenty-five stars. How big a T could I make? Would I have some stars left over?

The focus of attention changes sharply here. Until now, students have not been asked to focus on the number of stars in each shape. Various strategies will emerge. Take time to elicit and discuss the different approaches.

Some students may suddenly start counting the number of stars in each T that they have drawn, which can lead to some interesting reasoning. For instance, a student or group of students who count the stars in the 4th T (getting thirteen) may use inappropriate proportional reasoning to decide that the 8th T would have twice as many stars (twenty-six). They might also conclude that there would be a star missing for the 8th form and, therefore, the biggest complete star would be the 7th. Stacey (1989) identified this "proportional reasoning" strategy as a major source of difficulties and errors in pattern work. This is a good occasion to deal with this error because students can easily draw and count the number of stars in the 7th and 8th T shapes. Other students might use a trial-and-error approach, drawing the stars until they get (or do not get) a T shape and using any of their pattern perceptions to decide which one it is. For example, "There are eight stars in each of the arms around the middle star, so it's the 8th."

4. How many stars does it take to make the 10th, the 58th, or the 100th figure?

Now that attention is focused on the number of stars in the T shapes, students can be encouraged to use several of the pattern perceptions to answer these questions and, finally, to give a rule for how many stars are in any given T shape. This covers ground that is similar to question 2, although for students the switch from expressing how to draw a particular T to how many stars are in a particular T shape makes quite different demands on them.

5. How many stars does it take to make the *n*th figure?

Once a general formulation has been made for the number of stars in any T shape, students are ready to respond to the request for a verbal and then a written expression for the number of stars in the nth T shape. Their written expressions will probably not respect convention and may contain some words. However, the important thing at this point is to elicit attempts at symbolic writing that allow the students to communicate their different ways of expressing their "seeings" and "sayings" and to come to some agreement on the meaning of each. A possible student response to this question for each of the pattern perceptions is presented in **figure 4**.

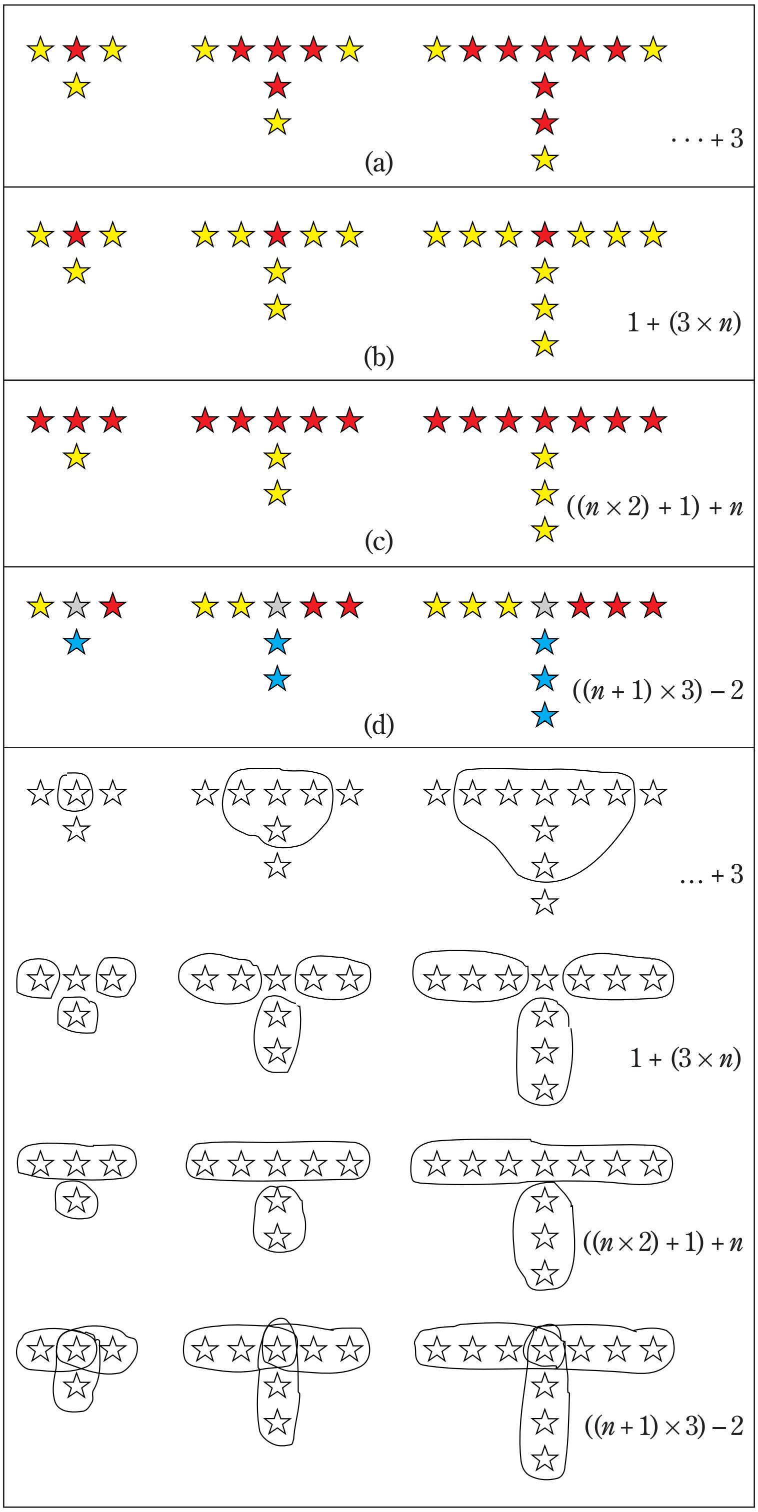


Fig. 4 **Different "sayings," or ways that the pattern could be described mathematically. The bottom half of the figure presents another mathematical view.**

6. Which of the expressions for the *n*th shape is a "right" one?

This question can lead to interesting work as students seek to justify each formula by checking to see if it works for the Ts that they can count or have already counted. This important step is often omitted by students when they are dealing with their own formulas.

If the first pattern perception has not already been put aside, it certainly will be here. It is a valuable lesson for students to note that some pattern perceptions are not easy to express symbolically. If students have developed some flexibility in pattern perception and are able to see the pattern in a number of ways, they will easily move to a different pattern perception that is more amenable to symbolic expression rather than locking on one inexpressible view and being unable to move.

Students will want to know how all these different-looking expressions can be correct. This situation opens the door for introducing equivalence of expressions. How, for example, can $1 + (3 \times n)$; $((n \times 2) + 1) + n$; and $((n + 1) \times 3) - 2$ all be equivalent to one another?

The difference between discussion around these particular expressions and a set of made-up expressions is at the belief and investment level. These expressions belong to the students; they express a whole world of shared meaning that has developed in the group. They own and care about these expressions and very much want to solve the dilemma that the expressions all appear to be correct, in spite of looking so different. They are thus motivated to engage in the next task of symbolic manipulation.

Here the teacher may up the ante by introducing a new rule or expression, $(n + n + 1) + n$, and asking if it, too, gives the number of stars in the *n*th T shape. This question may be followed with "What new 'seeing' does this express?"

The symbolic manipulation can be launched by asking students to think about how natural numbers behave when we operate on them (commutativity, associativity, distributivity) and to apply this knowledge to show that each expression is equivalent to $3n + 1$. Some conventions may be introduced as the work progresses (such as $2n$ for $n + n$ or $n \times 2$, the use of brackets to keep parts of expressions together, etc.). After satisfying themselves as to the equivalence of their expressions, students might be given other expressions and asked to verify their equivalence to $3n + 1$ and to determine to what pattern perception they might correspond. For example, $(1 + (n \times 4)) - n$ might be seen as a cross with the top removed or as a central dot with four arms from which the top arm of n stars is removed (see **fig. 5**).

In all this work, n has been a variable, any number in the natural numbers, and our focus has been on writing and verifying equivalent expressions. We have been laying the foundations for the study of relations and functions. A simple question can shift us into thinking of n as an unknown specific number and eliciting that other side of elementary algebra where we are concerned with finding unknowns through setting up and solving equations.

7. Which figure has exactly one hundred stars in it? What about fifty stars?

This inquiry takes us back to question 3's twenty-five-star problem that was used as a transition to get students to focus on the number of stars in each T shape. Here it is used to shift to the use of the letter as an unknown and the resultant activity of solving equations. The question here is this: "For what n can any of these equivalent expressions equal one hundred?" Students may want to reason from their various pattern perceptions, at first. For example, if $1 + 3n$ is to be equal to 100, then the three arms of n stars must contain ninety-nine stars. This means each arm contains thirty-three stars.

"Marlowe told me that she counted fifty stars in one of the Ts, and I responded that I think she miscounted. Why?" Here the teacher is asking if $1 + 3n$ can equal 50. This questioning can be extended in several directions, including the nature of the numbers that would correspond to a complete T shape.

8. How many interesting pattern problems can you create for the rest of the class?

After experience taking this questioning through a variety of patterns, students can now be asked to work in teams to create a pattern problem for the class. Asking students to create a pattern forces a synthesis of the activity: What is the nature of the activity we have been doing? What is a pattern? What are the different ways that people might see it and, therefore, express it? How do we decide (fairly quickly) if an expression is equivalent? What numbers of stars will make a form, and what numbers are excluded? How can we decide quickly? Here the students make the entire activity—not just the expressions—their own.

Closing Remarks

PATTERN WORK IS NOW UNDERTAKEN AS EARLY AS kindergarten, and both researchers and teachers have discovered that children engage in pattern work with great enthusiasm and innate ability. Having some flexibility in pattern perception and selecting mathematically useful patterns require some training, although there is nothing particularly algebraic about these skills. They are, however, a prerequisite to using patterns for introducing more formal algebra. Expressing pattern "seeings," saying a pattern with words and eventually using algebraic symbolism, is part of algebra. A pattern rule involves the use of a variable quantity. Although the variable in school pattern work is restricted to natural numbers, students can acquire some familiarity with the role of variables. Guided by scaffolded questioning, pattern explorations can lead to some very rich algebraic thinking about variables and unknowns, equivalence of algebraic expressions, symbol manipulation, domain and range of expressions and equations, and solving for the unknown.

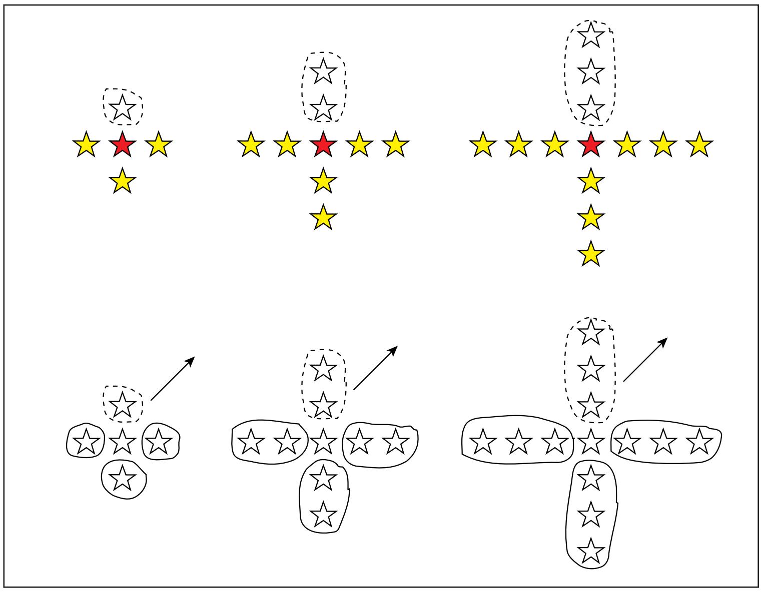


Fig. 5 One way of "seeing" $(1 + (n \times 4)) - n$

References

Lee, Lesley. "An Initiation into Algebraic Culture through Generalization Activities." In *Approaches to Algebra: Perspectives for Research and Teaching,* edited by Nadine Bednarz, Carolyn Kieran, and Lesley Lee, pp. 65–86. Dordrecht, The Netherlands: Kluwer, 1996.

Lee, Lesley, and David Wheeler. *Algebraic Thinking in High School Students: Their Conceptions of Generalization and Justification* (Research Report). Montréal, Canada: Concordia University, Mathematics Department, 1987.

National Council of Teachers of Mathematics (NCTM). *Principles and Standards for School Mathematics.* Reston, VA: NCTM, 2000.

Orton, Anthony, and Jean Orton. "Pattern and the Approach to Algebra." In *Pattern in the Teaching and Learning of Mathematics,* edited by A. Orton, pp. 104–20. London, UK: Cassell, 1999.

Stacey, Kaye. "Finding and Using Patterns in Linear Generalizing Problems." *Educational Studies in Mathematics* 20 (1989): 147–64. □